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FORMAL THEORY OF NOISY SENSOR NETWORK LOCALIZATION*

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Abstract. Graph theory has been used to characterize the solvability of the sensor network localization problem. If sensors correspond to vertices and edges correspond to sensor pairs between which the distance is known, a significant result in the theory of range-based sensor network localization is that if the graph underlying the sensor network is generically globally rigid and there is a suitable set of anchors at known positions, then the network can be localized, i.e., a unique set of sensor positions can be determined that is consistent with the data. In particular, for planar problems, provided the sensor network has three or more noncollinear anchors at known points, all sensors are located at generic points, and the intersensor distances corresponding to the graph edges are precisely known rather than being subject to measurement noise, generic global rigidity of the graph is necessary and sufficient for the network to be localizable (in the absence of any further information). In practice, however, distance measurements will never be exact, and the equations whose solutions deliver sensor positions in the noiseless case in general no longer have a solution. This paper then argues that if the distance measurement errors are not too great and otherwise the associated graph is generically globally rigid and there are three or more noncollinear anchors, the network will be approximately localizable, in the sense that estimates can be found for the sensor positions which are near the correct values; in particular, a bound on the position errors can be found in terms of a bound on the distance errors. The sensor positions in this case can be found by minimizing a cost function which, although nonconvex, does have a global minimum.

Key words. sensor networks, fault tolerance, inaccurate sensors, localization

AMS subject classifications. 05C90, 94C15, 49K40

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1. The problem of interest.

Consider a two-dimensional sensor network containing a set of nodes $S = \{s_1, s_2, \ldots, s_N\}$, and let distances d_{ij} between certain pairs of nodes s_i, s_j be given. Suppose further that the coordinates of certain nodes (termed the anchor nodes) are given. The localization problem is one of finding a map $\bar{p} : S \to \mathbb{R}^2$ which assigns coordinates $\bar{p}(s_i) \in \mathbb{R}^2$ to each node s_i such that, first, the assignments are consistent with the anchor node position data, and second, $||\bar{p}(s_i) - \bar{p}(s_j)|| = d_{ij}$ holds for all pairs i, j for which d_{ij} is given. One way localization can be thought about is in terms of solving multivariable polynomial equations. Let us write $\bar{p}(i)$ as shorthand for $\bar{p}(s_i)$. In particular, if $p(i)$ is used to denote a variable or unknown position for the node s_i, the values $\bar{p}(i)$ for $i = 1, 2, \ldots, N$, which are the true positions of the sensors are the solutions of the set of equations $||p(i) - p(j)|| = d_{ij}$

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when the d_{ij} are known data, with the additional constraints that $p(k) = \bar{p}(k)$ whenever s_k is designated as an anchor node, (for which $\bar{p}(k)$ is known a priori).

In this paper, we focus on the question of what happens when the distance data are noisy, i.e., subject to some error, presumably on account of the measurement process: instead of having available the true values d_{ij} associated with certain node pairs, we have quantities which are perturbations of these values. Now in the noiseless case when the relevant graph theoretic conditions are fulfilled which guarantee (unique) localizability, the set of equations $||p(i) - p(j)|| = d_{ij}$ reflecting the edge lengths (and with the anchor constraints $p(k) = \bar{p}(k)$ whenever s_k is designated as an anchor node) turns out to be an *overdetermined* set of equations (a fact which will be established in a later section), i.e., there are fewer scalar unknowns to be found than there are equations. Hence, when the quantities d_{ij} are perturbed from their true values, there will, in general, be no solution to the equations for the unknown $p(i)$. This is a complication to the problem of determining the $\bar{p}(i)$. It raises two questions of *what is meant by localization* in the noisy case and to what extent can one hope to obtain some kind of estimates for the sensor positions which are close to the true values, at least if the noise in the distance measurements is small. These are the central problems this paper considers.

Although recently many papers have considered the problem of localization using distance measurements in the presence of noise and proposed algorithmic solutions to solve the localization problem, see, e.g., [15, 4, 14, 3], none of them has formally studied when and how the problem of localization continues to be well-posed in the presence of noise.

In this paper, we will argue that, given the graph theoretic conditions that would guarantee unique localizability in the noiseless case, localization in the noisy case can be posed as a minimization problem, the solution of which has several properties. First, if the data are noiseless, the correct sensor positions are returned. Second, when the noise is not great, the solution of the minimization problem is unique and returns sensor position estimates which are not far from the correct values. Third, the errors between the true sensor positions and the estimates returned by solving the minimization problem go to zero continuously as the noise perturbations in the true distances go to zero. On the other hand, the minimization problem is not guaranteed to have just one local minimum, and the determination of an effective robust algorithm or numerical procedure for actually solving the problem to achieve localization, as for the noiseless case, is a separate and challenging task. While these results are not especially surprising, they serve to fill a logical gap between the formal treatment of noiseless localization and typical practical approaches to sensor localization in nonideal circumstances.

The next section provides background on the graph theoretic issues which underpin examination of the various equations and associated minimization problem. The third section states and proves the main result, while the fourth section contains brief concluding remarks.

2. Global rigidity and rigidity.

2.1. Combinatoric characterizations and properties. We associate a graph $G = (V, E)$ with a two-dimensional sensor network in the usual way. Each vertex of the graph corresponds to one sensor, and there is an edge in the graph joining two vertices just when the distance between the corresponding nodes of the sensor network is known.

In other work [2, 8], it has been shown that the (noiseless version of the) sensor network localization problem in $\mathbb{R}^2$ is, in principle, solvable if and only if the graph G

has the property of *generic global rigidity* [11] and there are at least three noncollinear anchor nodes. (The property of generic global rigidity, and the associated concept of global rigidity of a network or formation, will be explained in greater detail.) The qualifying words "in principle" imply that no consideration is being given at this point to the nature of a particular algorithmic procedure for carrying out the localization calculation. The word "generic" also deserves comment. Others sometimes remove the word generic from the term generic global rigidity with mild abuse of nomenclature, as, for example, in [11], where the term "global rigidity" is used instead of "generic global rigidity" as a descriptor of certain graphs. The term "uniquely realizable" has also been used for generic global rigidity; see [6, 11]. It turns out that there can be special networks which have at least three noncollinear anchors but which are not localizable although the associated graphs are globally rigid. They are exceptional, and in them, special relationships exist among the sensor positions, e.g., groups of sensors may be collinear. Thus the reason for using the term generic is to highlight the need to exclude the problems arising from such networks. For further discussion on the use of the word generic in the terminology generic global rigidity and generic rigidity, one may refer to [10, 13, 19].

Much of what follows depends on an understanding of the global rigidity concept and also the concept of rigidity. Therefore, before going further, we include some background information on these graph theory concepts. Let us call a *framework* a graph $G = (V, E)$ together with a map $\bar{p} : V \to \mathbb{R}^2$. Then $\bar{p}(v_i)$, written with mild abuse of notation as $\bar{p}(i)$, denotes the coordinate vector associated with vertex $v_i \in V$. Suppose a set of positive real numbers (representing intersensor distances) $D = \{d_{ij} : \{i, j\} \in E\}$ is defined. The framework is a *realization* if it results in $||\bar{p}(i) - \bar{p}(j)|| = d_{ij}$ for any $\{i, j\} \in E$. The two frameworks $(G, \bar{\mathbf{p}})$ and $(G, \tilde{\mathbf{p}})$ are *equivalent* if $||\bar{p}(i) - \bar{p}(j)|| = ||\tilde{p}(i) - \tilde{p}(j)||$ for any $\{i, j\} \in E$. The two frameworks $(G, \bar{\mathbf{p}})$ and $(G, \tilde{\mathbf{p}})$ are *congruent* if $||\bar{p}(i)-\bar{p}(j)|| = ||\tilde{p}(i)-\tilde{p}(j)||$ for all pairs i, j whether or not $\{i, j\} \in E$. This is equivalent to saying that $(G, \bar{\mathbf{p}})$ can be obtained from $(G, \tilde{\mathbf{p}})$ by an isometry of $\mathbb{R}^2$, i.e., a combination of translation, rotation, and reflection.

A framework is rigid when it cannot flex, i.e., it cannot via continuous motions respecting the edge constraints become noncongruent to its starting position. More precisely, $(G, \bar{\mathbf{p}})$ is rigid if there exists some positive ϵ such that if $(G, \bar{\mathbf{p}})$ and $(G, \tilde{\mathbf{p}})$ are equivalent and $||\bar{p}(i) - \tilde{p}(i)|| < \epsilon$ for all $i \in V$, then the two frameworks are congruent. It is important to understand, and we will use this fact below, that there exist rigid frameworks $(G, \bar{\mathbf{p}})$ and $(G, \tilde{\mathbf{p}})$ which are equivalent but not congruent [11]. In more detail, call a framework *minimally rigid* when the framework is rigid but the deletion of any single edge from the associated graph results in a nonrigid framework. Then any minimally rigid framework with more than three vertices is equivalent to another such framework to which it is not congruent. (Many rigid frameworks which are not minimally rigid have this property also.) Processes using concepts termed *flip ambiguity* and *(discontinuous) flex ambiguity* can always be used to construct from any given minimally rigid framework with four or more vertices an equivalent but noncongruent framework [9]; see Figure 2.1 for an illustration.

A framework $(G, \bar{\mathbf{p}})$ is *globally rigid* when every framework equivalent to $(G, \bar{\mathbf{p}})$ is also congruent to it.

Rigidity and global rigidity for a framework in $\mathbb{R}^2$ are generic properties, in the sense that if a framework $(G, \bar{\mathbf{p}})$ has either of these properties, then the framework $(G, \bar{\mathbf{p}})$ will also have the property for generic values of the position coordinates $\bar{\mathbf{p}}$, i.e., for all values save possibly those contained in a set involving an algebraic dependence over the rationals of the coordinates. Hence, not surprisingly, the properties of rigidity

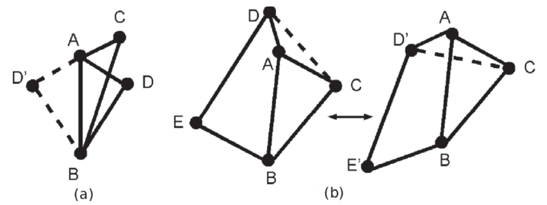

FIG. 2.1. *Illustration of* (a) *Flip ambiguity: Vertex D can be flipped over the edge (A, B) to a symmetric position D', and the distance constraints remain the same.* (b) *Discontinuous flex ambiguity: Temporarily removing the edge (C, D), the edge triple (D, A), (D, E), (E, B) can be flexed to obtain positions E' and D', such that the edge length (C, D) equals the edge length (C, D'), and then all the distance constraints are the same.*

and global rigidity can be cast purely in terms of properties of the underlying graph G, to which the terms generic rigidity and generic global rigidity can be applied.

A combinatoric necessary and sufficient condition for generic rigidity is encapsulated in Laman's theorem [12].

THEOREM 2.1 (Laman's theorem). *A graph $G = (V, E)$ modeling a framework in $\mathbb{R}^2$ of $|V|$ vertices and $|E|$ edges is generically rigid if and only if there exists a subgraph $G' = (V, E')$ with $2|V| - 3$ edges such that for any subset V'' of V, the induced subgraph $G'' = (V'', E'')$ of G' obeys $|E''| \leq 2|V''| - 3$.*

A necessary condition for generic rigidity is evidently that there are at least $2|V| - 3$ edge constraints. A necessary and sufficient condition for generic global rigidity is that G remains generically rigid when any edge is removed and that between any two vertices, there exist at least three paths which are nonintersecting except at the two vertices in question, i.e., G is 3-*connected* [7]. Hence, in a generically global rigid graph, there are necessarily at least $2|V| - 2$ edges.

It is clear that *a framework is an abstraction of a sensor network*. However, even given the graph and distance set of a globally rigid framework, there is not enough information to position the framework absolutely in $\mathbb{R}^2$. In fact, as noted above, the framework can be positioned only to within a translation, rotation, or reflection. To eliminate this nonuniqueness requires further knowledge, typically the absolute position of at least three vertices. In a physical sensor network, this information is either derived from global position sensing measurements or other form of independent measurements. The vertices in question must not be collinear, for if they were, there would be ambiguity up to a reflection in the position of all other vertices.

2.2. Linear algebra characterization and properties. There is a different straightforward characterization of rigidity for a framework in linear algebra terms, using the concept of the rigidity matrix.

Consider a graph $G = (V, E)$ modeling a framework in $\mathbb{R}^2$ of $|V|$ vertices and $|E|$ edges. Let the coordinate vector $\bar{p}(j)$ of vertex v_j be $[x_j, y_j]^\top$. The rigidity matrix is defined with an arbitrary ordering of the vertices and edges and has $2|V|$ columns and $|E|$ rows. Each edge gives rise to a row, and if the edge links vertices j and k, the nonzero entries of the row of the matrix are in columns $2j - 1$, $2j$, $2k - 1$, and $2k$ and are, respectively, $x_j - x_k$, $y_j - y_k$, $x_k - x_j$, $y_k - y_j$. For example, for the graphs of Figures 2.2(a) and (d), the rigidity matrices are

$$R = \begin{bmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 - x_4 & y_3 - y_4 & x_4 - x_3 & y_4 - y_3 \\ x_1 - x_4 & y_1 - y_4 & 0 & 0 & 0 & 0 & x_4 - x_1 & y_4 - y_1 \end{bmatrix}$$

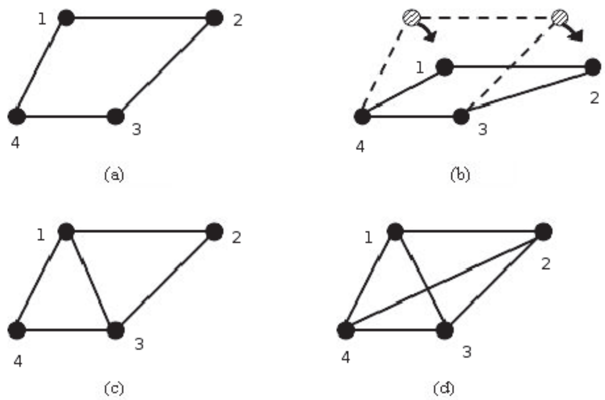

FIG. 2.2. *Rigid and nonrigid formations. The formation represented in* (a) *is not rigid. It can be deformed by a smooth motion without affecting the distance between the agents connected by the edges, as shown in* (b). *The formations represented in* (c) *and* (d) *are rigid, as they cannot be deformed by any such move. In addition, the formation represented in* (c) *is minimally rigid because the removal of any edge would render it nonrigid. That of* (d) *is not minimally rigid; any edge may be removed without losing rigidity.*

and

$$R = \begin{bmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 - x_4 & y_3 - y_4 & x_4 - x_3 & y_4 - y_3 \\ x_1 - x_4 & y_1 - y_4 & 0 & 0 & 0 & 0 & x_4 - x_1 & y_4 - y_1 \\ x_1 - x_3 & y_1 - y_3 & 0 & 0 & x_3 - x_1 & y_3 - y_1 & 0 & 0 \\ 0 & 0 & x_2 - x_4 & y_2 - y_4 & 0 & 0 & x_4 - x_2 & y_4 - y_2 \end{bmatrix}.$$

The key result connecting generic rigidity of a graph and the rigidity matrix is shown in the following theorem [19].

THEOREM 2.2. *A graph $G = (V, E)$ modeling a framework in $\mathbb{R}^2$ of $|V|$ vertices and $|E|$ edges is generically rigid if and only if for generic vertex positions (or at least one set of vertex positions), the rigidity matrix has rank $2|V| - 3$.*

It is easy to verify this result for the first example at least. With the generic vertex position, the rank for the rigidity matrix R of Figure 2.2(a) is 4, and for Figures 2.2(c) and (d), it is 5. At some special positions for the vertices of a framework, the rigidity of the latter graph may be lost, often when vertices are collinear in $\mathbb{R}^2$. If all vertices are collinear in Figure 2.2(d), for example, column $2j$ of R is a multiple of column $2j-1$ for each j and so the rank of R is at most 4. Such situations are nongeneric.

There is intuitive content associated with the kernel of the rigidity matrix. In case the rigidity matrix has rank $2|V| - 3$, the dimension of the kernel is 3. Any vector in the kernel then corresponds to velocities for the vertices of the framework when the framework is translating and/or rotating. (In two dimensions, there are two independent translations and one rotation that are always possible.) When the kernel dimension is greater than 3, independent motions in addition to translation and rotation are possible, corresponding to some kind of flexing. In more detail, suppose that $\{i, j\} \in E$, so that the coordinates of vertices v_i, v_j in the framework obey for all time

$$||\bar{p}(i,t) - \bar{p}(j,t)||^2 = d_{ij}^2. \qquad (2.1)$$

This equation captures the notion that the distance between the two vertices takes a fixed value, the square of the value being d_{ij}^2. Assuming the motion is smooth, it follows that

$$[\bar{p}(i,t) - \bar{p}(j,t)]^\top \frac{d}{dt}[\bar{p}(i,t) - \bar{p}(j,t)] = 0, \tag{2.2}$$

and by stacking together $|E|$ such equations, there results

$$R\frac{d}{dt}\bar{\mathbf{p}}(t) = 0, \tag{2.3}$$

where $\bar{\mathbf{p}}(t)$ denotes the $2|V|$-vector obtained by stacking the $\bar{p}(i,t)$.

There is a second useful consequence of (2.1). Suppose that vertex positions are initially fixed but then a small displacement $\delta\bar{\mathbf{p}}$ is made to the $2|V|$-vector of vertex positions *without* respecting the length constraints. There will, of course, be a corresponding change in the lengths corresponding to the edges of the graph. *To first order*, this change is described by

$$\delta\mathbf{d} = 2R\delta\bar{\mathbf{p}}, \tag{2.4}$$

where $\delta\mathbf{d}$ is the vector of changes in *the squares of* the lengths, ordered in the same way as the edges are ordered in defining R.

Because R is not square, it is not invertible. Therefore, it does not immediately make sense to contemplate the change in vertex positions that would flow from an arbitrary, even small, change in lengths, at least without some kind of constraint. One can, however, contemplate constraining some of the vertices not to move and some of the lengths not to change. Then a submatrix of R will map small changes in some of the vertex positions to small changes in some of the squares of the lengths, and if the matrix should be square and nonsingular, the inverse of this matrix will map small changes in some of the squares of the lengths to small changes in some of the vertex positions.

3. Equation sets and minimization problems for localization.

3.1. The overdetermined property of the equation set. Regarding $p(i)$ as a 2-vector corresponding to the unknown position of vertex v_i, the equations which apply to a framework are, in terms of the notation above,

$$||p(i) - p(j)||^2 = d_{ij}^2 \ \forall \{i,j\} \in E. \tag{3.1}$$

One such equation is associated with each prescribed intersensor distance. (Of course, we could have written the equations without the squares, but then they would no longer be polynomial in the variables.) A particular set of values $\bar{p}(i), i = 1, 2, \ldots, |V|$ satisfying these equations provides a particular set of vertex coordinates for the framework vertices consistent with the distance constraint set.

A modified set of equations applies when the framework is derived from a sensor network where there are anchor nodes since some of the node positions are now known. Write the set of vertices of the associated graph as $V = V_O \cup V_A$, where V_A comprises precisely vertices corresponding to the anchor nodes, of which there must be at least three, and V_O is the set of (ordinary) vertices which do not correspond to anchor nodes. Let the coordinate values for the anchor nodes be $\bar{p}(i)$ for $i \in V_A$. Between any two anchor nodes of the network, the distance is necessarily known. Denote by

E_A the subset of E comprising those edges joining two vertices which correspond to anchor nodes.

Then the equations which apply to the framework after using the anchor node information include distance information *and* coordinate information and are of the form

$$||p(i) - p(j)||^2 = d_{ij}^2 \ \forall \{i, j\} \in E \setminus E_A, \quad p(i) = \bar{p}(i) \ \forall i \in V_A. \tag{3.2}$$

(Note that we have discarded the equations corresponding to the length constraint applying to the distance between two anchor nodes in formulating this equation set, i.e., the edges E_A, since such equations involve no unknown quantities.) Determining a set of values $\bar{p}(i)$ for all $i \in V_O$ satisfying these equations is the localization problem, at least when measurement data are exact.

We shall now establish that for a globally rigid framework, the number of independent equations actually exceeds the number of (scalar) unknowns we are seeking to determine, i.e., the set is overdetermined.

LEMMA 3.1. *Consider a globally rigid framework $F = (G, \bar{\mathbf{p}})$ with $G = (V, E)$. Suppose the vertex set V can be partitioned as $V = V_O \cup V_A$, where $|V_A| \geq 3$ and the values $\bar{p}(i)$ are known for all $v_i \in V_A$ and unknown for all $v_i \in V \setminus V_A$. Then the number of edge-length constraint equations in the equation set* (3.2) *exceeds* $2|V_O|$.

Proof. Since the framework is globally rigid, we can drop any edge and it remains rigid. Choose the dropped edge to be one of those in the edge-length constraint set appearing in (3.2), i.e., not in E_A. Since the associated graph is now rigid, there exists by Laman's theorem a minimally rigid subgraph $G' = (V, E')$, and thus one with $|E'| = 2|V| - 3$ edges, such that any induced subgraph of G' defined using any subset V'' of V has at most $2|V''| - 3$ edges. Choose for V'' the set V_A. Then in G', there are at most $2|V_A| - 3$ edges joining vertex pairs in V_A and so at least $[2|V| - 3] - [2|V_A| - 3] = 2|V_O|$ other edges. Hence, apart from the edge that was dropped from the edge constraint set $E \setminus E_A$ before constructing G', there are necessarily at least $2|V_O|$ distance constraint equations in (3.2), i.e., at least $2|V_O| + 1$ distance constraint equations in all. There are also precisely $2|V_O|$ unknowns to be determined from the equation set (3.2), taking the coordinates of the anchor nodes as known. Therefore, the unknown coordinates are the solutions of an *overdetermined* set of equations. ∎

The equation set (3.2) constitutes a set of simultaneous polynomial equations in the position coordinates of the nonanchor nodes. In general, simultaneous polynomial equations have multiple solutions (unless the equations are linear); however, an overdetermined set, if it has a solution, can be such that the solution is unique.

A very simple example of the above formulation arises in considering the localization of one sensor given its distance from each of three anchors when these have known positions and are not collinear. Using two of the distance measurements, the sensor with unknown position can be determined to within a binary ambiguity, given by the intersection of two circles. The binary ambiguity is resolved using the third distance measurement; the sensor with unknown position is at the single common point of intersection of three circles with known centers and radii. (The fact that there is a single common point of intersection is a consequence of the noncollinearity of the anchors.) The number of scalar unknowns is two, being the two coordinates of the sensor with unknown position. The number of distance equations is three.

3.2. Posing a noisy localization problem. Another feature of the overdetermined nature of the equations becomes evident when we postulate errors in the (squares of the) distance measurements. Suppose that each squared distance d_{ij}^2 in (3.2) is replaced by $d_{ij}^2 + n_{ij}$, the quantity n_{ij} being a (typically small) error in the squared distance (rather than in the distance itself); thus d_{ij} remains the actual distance, and n_{ij} constitutes the measurement noise effect. Then in the absence of any knowledge of the noise, it is natural to consider the following set:

$$\begin{aligned} ||p(i) - p(j)||^2 &= d_{ij}^2 + n_{ij} \ \forall \{i,j\} \in E \setminus E_A, \\ p(i) &= \bar{p}(i) \ \forall i \in V_A. \end{aligned} \tag{3.3}$$

This equation set is still overdetermined but in general will have no solution. The fact that it has no solution underpins the motivation for the paper.

The simplest example of this problem involves localizing a single sensor given noisy measurements of its distance from three anchor sensors, as treated in [5]. As already noted, there are two unknowns, the coordinates of the single sensor to be localized. There are three scalar equations perturbed by noise, and there is generically no solution. The obvious remedy is to try for an approximate solution, and that is what is done in general.

Despite the inability to solve the noisy equation set (3.3), the notion of localization, albeit approximate localization, still makes sense: clearly, it would be appropriate to seek those coordinate values of $p(i)$, call them $\bar{p}^*(i)$ for $i \in V_O = V \setminus V_A$ solving the following minimization problem:

$$\begin{gathered} \min_{p(i), i \in V_O} \sum_{\{i,j\} \in E \setminus E_A} [||p(i) - p(j)||^2 - (d_{ij}^2 + n_{ij})]^2 \\ \text{subject to} \\ p(i) = \bar{p}(i) \ \forall i \in V_A. \end{gathered} \tag{3.4}$$

(Of course, other measures for the error between $||p(i) - p(j)||^2$ and $(d_{ij}^2 + n_{ij})$ could be used. There will be no essential difference in the results.) Now we know that if all n_{ij} are zero, there is generically a unique solution to the minimization problem, namely, the solution of the usual localization problem, which yields a zero value for the cost function. Let $\mathbf{n}$ denote the vector of n_{ij}, corresponding to some arbitrary ordering of the subset of edges $E \setminus E_A$, i.e., edges incident on at least one ordinary (nonanchor) vertex. Let $||\mathbf{n}||$ denote the Euclidean norm so that $||\mathbf{n}||^2 = \sum_{\{i,j\} \in E \setminus E_A} n_{ij}^2$. The questions of interest to us here are: *Is it guaranteed that, at least for suitably small* $||\mathbf{n}||$*, there will still be a solution to the minimization problem* and *that its distance from the solution of the localization problem with zero measurement noise will go continuously to zero as the value of* $||\mathbf{n}||$ *goes to zero?* A further question is: *Under what circumstances will the solution of the (noisy) minimization problem be unique?*

Before stating the main result, let us motivate the need to limit $||\mathbf{n}||$ or equivalently the magnitude of the $|n_{ij}|$. Consider Figure 3.1, which represents a sensor network with four nodes in two configurations, one corresponding to nodes 1, 2, 3, and 4 and the other corresponding to nodes 1′, 2, 3, and 4. Suppose that nodes 2, 3, and 4 are anchors, and regard 1 as being in a true position. It is quite apparent from the figure that the distances d_{12}, d_{13}, d_{14} are close to the distances $d_{1'2}, d_{1'3}, d_{1'4}$. If there are noisy measurements of d_{12}, d_{13}, d_{14} and the measurement error magnitudes are comparable to the differences between $d_{12}, d_{1'2}$, etc., i.e., if the noisy measurements of d_{12}, d_{13}, d_{14} yield values approximately equal to $d_{1'2}, d_{1'3}, d_{1'4}$, then the solution of the minimization problem could well give a point in the vicinity of 1′ rather than in the vicinity of 1. On the other hand, there will be a value of measurement error such that if it is not exceeded, there could be no possibility of this occurring.

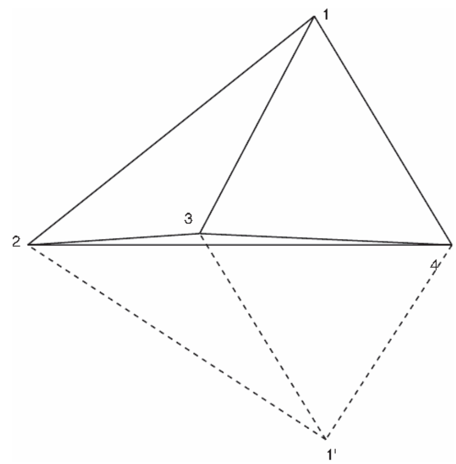

FIG. 3.1. *A four-node sensor network in two configurations, one corresponding to nodes* 1, 2, 3, *and* 4 *and the other corresponding to nodes* $1', 2, 3,$ *and* 4.

3.3. Main result. The central result of this paper is the following theorem.

THEOREM 3.2. *Consider a globally rigid and generic framework F defined by a graph $G = (V, E)$ and vertex positions $\bar{p}(i), i = 1, 2, \ldots, |V|$. Denote the formation as $F = F(V, E, \bar{\mathbf{p}})$. Let $V_A \subset V$ denote vertices of G corresponding to anchor nodes, of which there are at least three and for which the value of $\bar{p}(i)$ is known, and let $E_A \subset E$ denote those edges incident on two vertices of V_A, with the graph $G_A = (V_A, E_A)$ then forming a complete subgraph of G. Let d_{ij} denote the distance between nodes i and j when ij is an edge of G. Consider the minimization problem* (3.4)*, and denote the solution of the minimization problem by $\bar{\mathbf{p}}^*$. Then there exists a suitably small positive Δ and an associated positive constant c such that if the measurement errors in the squares of the distances obey $||\mathbf{n}|| < \Delta$, the solution of the minimization problem is unique and there holds $||\bar{\mathbf{p}}^* - \bar{\mathbf{p}}|| \leq c||\mathbf{n}||$.*

The issue of what determines Δ and c and whether they are readily computed or estimated will be dealt with subsequently.

We shall use several lemmas in proving the theorem. Some of these appear to be of independent interest.

3.4. Background lemmas. For convenience, assume that the vertices in the set $V \setminus V_A = V_O$ are indexed from 1 to $K := |V_O|$. Recall that we are seeking to minimize the following cost function:

$$P[p(1), p(2), \ldots, p(K)] = \sum_{ij \in E \setminus E_A} [||p(i) - p(j)||^2 - (d_{ij}^2 + n_{ij})]^2 \tag{3.5}$$

subject to

$$p(i) = \bar{p}(i) \text{ for } i = K + 1, K + 2, \ldots, |V|. \tag{3.6}$$

For fixed $\mathbf{n}$, P is a function of the vector $\mathbf{p}$ or, equivalently, $x_1, y_1, x_2, y_2, \ldots, y_n$.

For the first lemma, we require some additional notation. Define the *reduced rigidity matrix* R_r to be the submatrix of R containing those columns corresponding to vertices $1, 2, \ldots, K$ and those edges joining vertex pairs of which at least one is in the set $1, 2, \ldots, K$. Note that the entries of R_r depend in an affine way on each of the x_i, y_i. Let $\mathbf{e}$ denote the vector of quantities $e_{ij} = ||p(i) - p(j)||^2 - d_{ij} - n_{ij}$ when $\{i, j\} \in E$ and with the same ordering as the rows of the rigidity matrix; the entries depend on the x_i, y_i and the n_{ij}. Define $e_{ij} = 0$ when $ij \notin E$, and let $\mathcal{E}$ denote the square $K \times K$ matrix with

$$\mathcal{E} = \begin{pmatrix} -\sum_j e_{1j} & e_{12} & e_{13} & \dots & e_{1K} \\ e_{12} & -\sum_j e_{2j} & e_{23} & \dots & e_{2K} \\ e_{13} & e_{23} & -\sum_j e_{3j} & \dots & e_{3K} \\ \vdots & \vdots & & & \vdots \\ e_{1K} & e_{2K} & e_{3K} & \dots & -\sum_j e_{jK} \end{pmatrix}. \tag{3.7}$$

Now using a straightforward calculation, we have the following lemma.

LEMMA 3.3. *Adopt the same notation as above. Then the column vector* ∇P *whose* $(2i-1)$*th and* $2i$*th entries are* $\frac{\partial P}{\partial x_i}$ *and* $\frac{\partial P}{\partial y_i}$ *is given by*

$$\nabla P = 2R_r^\top \mathbf{e}, \tag{3.8}$$

where the reduced rigidity matrix R_r *is evaluated at the* $p(i), i = 1, 2, \dots |V|$. *Further, the Hessian matrix* $\nabla^2 P$ *is given by*

$$\nabla^2 P = 2[R_r^\top R_r + \mathcal{E} \otimes I_2]. \tag{3.9}$$

The matrix R_r plays a crucial role in the above formulas and in the material to come. As already recorded (see Theorem 2.2), for a rigid framework, the rigidity matrix R must have rank $2|V| - 3$. We will need the rank of R_r rather than R. Its rank is given in the following lemma; also see [18].

LEMMA 3.4. *Adopt the notation above, with the assumption that the framework* $F(V, E, \bar{\mathbf{p}})$ *is globally rigid and that a subset* $V_A \subset V$ *of vertices are designated as anchor vertices, with an edge existing between every pair of vertices in* V_A. *Then the reduced rigidity matrix* R_r *(evaluated at* $\bar{\mathbf{p}}$*) has generically full column rank.*

Proof. The rigidity matrix R has $|E|$ rows and $2|V|$ columns and is of rank $2|V|-3$. Suppose that the vertices are ordered so that the last numbered vertices correspond to V_A and that the edges are ordered so that the edges joining vertices both in V_A appear last. (There are $\frac{1}{2}|V_A|(|V_A| - 1)$ such edges, and since $|V_A| \geq 3$, there are necessarily at least 3.) Then it is straightforward to see that for some matrices X and Y with $2V_A$ columns and with Y with at least $\frac{1}{2}|V_A|(|V_A| - 1)$ rows, there holds

$$R = \begin{bmatrix} R_r & X \\ 0 & Y \end{bmatrix}. \tag{3.10}$$

Now if R_r were not of full column rank, there would be a nonzero vector, call it α, such that $R_r\alpha = 0$. We shall obtain a contradiction. Define $\beta = [\alpha\top \;\; 0]^\top$. Then $R\beta = 0$.

Now it is known from [19] that because R is the rigidity matrix of a rigid framework, its kernel is three-dimensional and the following three vectors are a basis of the kernel: $\lambda_1 = [1\ 0\ 1\ 0\ \dots]^\top$, $\lambda_2 = [0\ 1\ 0\ 1\ \dots]^\top$, and $\lambda_3 = [y_1\ \ -x_1\ y_2\ \ -x_2\ \dots]^\top$. Here $[x_i\ y_i]^\top$ denotes the coordinate vector of the ith vertex. Hence β must be a linear combination of λ_1, λ_2, and λ_3, i.e., for some scalar a, b, and c, not all zero:

$$\beta = a\lambda_1 + b\lambda_2 + c\lambda_3. \tag{3.11}$$

The last $\frac{1}{2}|V_A|(|V_A|-1)$ rows give

$$a\bar{\lambda}_1 + b\bar{\lambda}_2 + c\bar{\lambda}_3 = 0, \tag{3.12}$$

where $\bar{\lambda}_1, \bar{\lambda}_2$, and $\bar{\lambda}_3$ are subvectors formed from the last $\frac{1}{2}|V_A|(|V_A|-1) \geq 3$ entries of λ_1, λ_2, and λ_3, respectively. However, inspection of the subvectors formed from the last three entries of $\bar{\lambda}_1, \bar{\lambda}_2$, and $\bar{\lambda}_3$ shows they are generically independent. Hence nonzero a, b, and c cannot exist, i.e., there is no nonzero α that lies in the kernel of R_r. ∎

The preceding two lemmas will now provide the basis for using the implicit function theorem to show that the noisy minimization problem has a *locally* minimizing solution when the noise is small enough.

Lemma 3.5. *Assume the hypotheses of Theorem* 3.2. *Consider the set of equations* $\nabla P = 0$*, which are necessarily satisfied at a minimum of the index* (3.5)*. Then the following are true:*

1. *These equations have the solution* $p(i) = \bar{p}(i), i = 1, 2, \ldots, K$ *when the* n_{ij} *for* $\{i, j\} \in E \setminus E_A$ *are all zero.*
2. *There exist a suitably small positive* Δ_1 *and a positive constant* c *depending on* Δ_1 *such that for any fixed* $\mathbf{n}$ *lying in the ball of radius* Δ_1 *around the origin, there is a unique solution* $\hat{p}(i), i = 1, 2, \ldots, K$ *of the equations* $\nabla P = 0$ *satisfying the constraint* $||\hat{\mathbf{p}} - \bar{\mathbf{p}}|| \leq c||\mathbf{n}||$.
3. *For the fixed* $\mathbf{n}$*, this solution is a local minimizer (with respect to* $\mathbf{p}$*) of* $P(\mathbf{p}, \mathbf{n})$.

Proof. The first conclusion of the lemma is trivial, amounting to the conclusion that because the network is globally rigid, the actual vertex positions will be recovered when measurements are noiseless.

To establish the second claim of the lemma, one can apply the implicit function theorem to the equation $\nabla P(\mathbf{p}, \mathbf{n}) = 0$. The second claim of the lemma will hold true, provided the Jacobian of ∇P is nonsingular at the solution point of $\nabla P = 0$ defined by the pair $(\bar{\mathbf{p}}, \mathbf{n}) = 0$. The Jacobian is precisely $\nabla^2 P$ which is evaluated in Lemma 3.3. Because by Lemma 3.4 R_r has full column rank, the matrix $R_r^\top R_r$ is positive definite and certainly nonsingular. Also, the matrix $\mathcal{E}$ is zero at this point. Hence the second formula of Lemma 3.3 yields that $\nabla^2 P$ is nonsingular, as required. For the third part, since at $\mathbf{n} = 0$, $\nabla^2 P$ is positive definite, the continuity of this expression in $\mathbf{n}$ guarantees it is positive definite for $||\mathbf{n}||$ suitably small, which means that the stationary point $\hat{\mathbf{p}}$ is minimizing. ∎

Lemma 3.5 shows that under the hypothesis of Theorem 3.2, with sufficiently small noise so that the inverse function theorem becomes applicable, a *local* minimum for $P(\mathbf{p}, \mathbf{n})$ is achieved with $\mathbf{p} = \hat{\mathbf{p}}$. To prove the theorem, it remains to show that this minimum is also a global minimum, i.e., that $\hat{\mathbf{p}}$ coincides with $\mathbf{p}^*$.

3.5. Proof of the theorem. Let $\mathcal{B}$ denote the ball around $\bar{\mathbf{p}}$ defined by $||\mathbf{p} - \bar{\mathbf{p}}|| < c\Delta_1$, and let $\mathcal{B}^c$ denote the complementary set $||\mathbf{p} - \bar{\mathbf{p}}|| \geq c\Delta_1$. Observe that for all $\mathbf{n}$ with $||\mathbf{n}|| < \Delta_1$, the immediately preceding lemma guarantees that $\hat{\mathbf{p}} \in \mathcal{B}$. Let P_1 be defined by

$$\begin{aligned} P_1 = \inf_{p(i), i \in V_O, \mathbf{p} \in \mathcal{B}^c} & \sum_{\{i,j\} \in E \setminus E_A} [||p(i) - p(j)||^2 - d_{ij}^2]^2 \\ & \text{subject to} \\ p(i) = \bar{p}(i)\ & \forall i \in V_A. \end{aligned} \tag{3.13}$$

The unique localizability property with zero noise, i.e., there exist unique $p(i)$ and $p(j)$ such that $||p(i) - p(j)|| = d_{ij}$, $i, j \in \{1, \ldots, N\}$, guarantees that P_1 is positive. Also, P_1 is overbounded by the minimum value of P computed on $||\mathbf{p} - \bar{\mathbf{p}}|| = c\Delta_1$.

Consider also a collection of minimization problems, parameterized by a nonnegative constant Δ_2, with variables $\mathbf{p}$ and $\mathbf{n}$:

$$\begin{aligned} P_2 = & \inf_{p(i), i \in V_O, \mathbf{p} \in \mathcal{B}^c, ||\mathbf{n}|| \leq \Delta_2} \sum_{\{i,j\} \in E \setminus E_A} [||p(i) - p(j)||^2 - (d_{ij}^2 + n_{ij})]^2 \\ & \text{subject to} \\ p(i) = & \bar{p}(i) \; \forall i \in V_A. \end{aligned} \tag{3.14}$$

There is an infimum as opposed to minimum used in (3.14) because the set over which the extremization is performed is unbounded. Let us now argue that a bounded set can be used, with a minimum resulting. With Δ_2 fixed, we claim that the value of P in (3.5) computed on the set $\{(\mathbf{p}, \mathbf{n}) : ||\mathbf{p} - \bar{\mathbf{p}}|| = R \wedge ||\mathbf{n}|| \leq \Delta_2\}$ for large enough R will go to infinity as $R \to \infty$: Suppose that for a fixed $\mathbf{n}$ obeying the constraint, $\mathbf{p_R}$ achieves the minimum of P on $||\mathbf{p} - \bar{\mathbf{p}}|| = R$, and suppose that for a particular R, the 2-vector position of, say, the Ith vertex $p_R(I)$ is of order R. Since the underlying graph is connected, there will be a path from any anchor node to node I, and so the length of at least one edge along this path will be of order R. Therefore the corresponding summand in (3.5) will be of order R^4.

It follows that the infimum P_2 of the index P over the set $\mathcal{B}^c \bigcap ||\mathbf{n}|| \leq \Delta_2$ is going to be attained over the intersection of the set $\mathcal{B}_R^c := \mathcal{B}^c \bigcap ||\mathbf{p} - \bar{\mathbf{p}}|| \leq R$ for some suitably large R and $||\mathbf{n}|| \leq \Delta_2$, and because this is a bounded and closed set, there will be at least one point in it achieving the minimum.

We shall now choose Δ_2. First note that when $\Delta_2 = 0$, there holds $P_2 = P_1$. Because P depends continuously on the n_{ij}, P_2 will depend continuously on Δ_2 as Δ_2 increases from 0. If $P_2 \geq (1/2)P_1$ for all $\Delta_2 \leq \Delta_1$, choose $\Delta_2 = \Delta_1$. Otherwise, choose Δ_2 so that $P_2 = (1/2)P_1$, which requires $\Delta_2 < \Delta_1$.

By the argument above, the global minimum of P over the set $\mathcal{B}^c$ for a fixed $\mathbf{n}$ obeying $||\mathbf{n}|| \leq \Delta_2$ is at least $(1/2)P_1$. By Lemma 3.5, there is a single local, and therefore global, minimum of P in the closure of the set $\mathcal{B}$ for any fixed $\mathbf{n}$ obeying $||\mathbf{n}|| \leq \Delta_1$ and a fortiori $||\mathbf{n}|| \leq \Delta_2$. If this minimum is less than $(1/2)P_1$, it is necessarily the global minimum with no restriction on the set of allowed $\mathbf{p}$. Define Δ by

$$\Delta = \min\{\Delta_2, [(1/2)P_1]^{1/2}\}. \tag{3.15}$$

Observe that for any fixed $\mathbf{n}$, there holds

$$P^* = \min_{p(i), i \in V_O} \sum_{\{i,j\} \in E \setminus E_A} [||p(i) - p(j)||^2 - (d_{ij}^2 + n_{ij})]^2 \tag{3.16}$$

$$\leq \sum_{\{i,j\} \in E \setminus E_A} [||\bar{p}(i) - \bar{p}(j)||^2 - (d_{ij}^2 + n_{ij})]^2 \tag{3.17}$$

$$= ||\mathbf{n}||^2. \tag{3.18}$$

Now require $||\mathbf{n}|| < \Delta$. Then this formula ensures that $P^* < (1/2)P_1$. Such a minimum cannot be achieved in the set $\mathcal{B}^c$. Hence it is achieved in the set $\mathcal{B}$. Within this set, there is at most one minimum. Therefore the minimum in this set is the global minimum with no restriction on the set of allowed $\mathbf{p}$. This proves the theorem.

3.6. Remarks on the constants in the theorem. The main theorem involves two constants c and Δ. This subsection makes some remarks on these constants.

The constant c arises in a more or less standard way in applying the inverse function theorem. Indeed, it is standard that in an infinitesimal neighborhood around $\bar{\mathbf{p}}, \bar{\mathbf{n}} = 0$, there holds

$$\nabla^2 P(\bar{\mathbf{p}}^* - \bar{\mathbf{p}}) = \left[\frac{\partial}{\partial \mathbf{n}}(\nabla P)\right]\bar{\mathbf{n}} = 2R_r^\top \mathbf{n}. \tag{3.19}$$

Noting the expression for $\nabla^2 P$ in (3.9) and that at $\bar{\mathbf{p}}$, one has $\mathcal{E} = 0$, we see that, infinitesimally,

$$R_r^\top R_r(\bar{\mathbf{p}}^* - \bar{\mathbf{p}}) = R_r^\top \mathbf{n} \tag{3.20}$$

so that

$$c = [\lambda_{\min}^{1/2}(R_r^\top R_r)]^{-1}. \tag{3.21}$$

In this equation, R_r is evaluated at $\bar{\mathbf{p}}$. As noted, the value of c applies for infinitesimally small perturbations of $\mathbf{n}$ around zero. It is, of course, a guide for larger perturbations. For infinitesimally small perturbations around a nonzero value of $\mathbf{n}$, there holds, using (3.19),

$$c = [\lambda_{\max}[R_r(\nabla^2 P)^{-2}R_r^\top]]^{1/2} = [\lambda_{\max}[(\nabla^2 P)^{-2}R_r^\top R_r]]^{1/2}. \tag{3.22}$$

In this equation, the various quantities are evaluated at the nominal value of $\mathbf{n}$ and the corresponding value of $\bar{\mathbf{p}}$. For larger perturbations with $\mathbf{n}$ restricted by $||\mathbf{n}|| < \Delta$, it then follows that

$$c = \max_{||\mathbf{n}||\leq\Delta}[\lambda_{\max}[R_r(\nabla^2 P)^{-2}R_r^\top]]^{1/2} = \max_{||\mathbf{n}||\leq\Delta}[\lambda_{\max}[(\nabla^2 P)^{-2}R_r^\top R_r]]^{1/2}. \tag{3.23}$$

The value of (3.21) is a lower bound to the value just obtained. If Δ is small, it may be a good approximation to the correct value of c.

Now we turn to explaining the value of Δ. There are two reasons why this quantity may be limited. First, the set over which the inverse function theorem is valid is necessarily one where $\nabla^2 P$ is positive definite. This property holds at the point where $\mathbf{n} = 0$. It may cease to hold well away from this point. It is this phenomenon which limits the size of Δ_1. There is a second independent limiting factor. In the noiseless case, there may be a local minimum for the function P differing from zero by a small amount. The coordinate values corresponding to this local but nonglobal minimum will necessarily be different from those corresponding to the global minimum. Then it is likely that as noise levels are slowly increased from zero, the coordinates yielding the global minimum could jump at some noise level from the vicinity of those applicable in the noiseless case to coordinates corresponding to a point in the vicinity of that associated in the noiseless case with the local but nonglobal minimum. To stop this from happening, it may be necessary to take $||\mathbf{n}|| \leq \Delta_2 < \Delta_1$ (this, in particular, ensures that the minimum value of the cost function in the vicinity of any differing local minimum does not get too small, in particular, not smaller than $(1/2)P_1$) and also to take $||\mathbf{n}|| < [(1/2)P_1]^{1/2}$ to ensure that the value of the noisy cost function at the minimum in the vicinity of the noiseless global minimum does not get too large, in particular, not greater than $(1/2)P_1$.

4. Conclusions. Sensor network localization in the noiseless case is a matter of finding a unique solution of some overdetermined equations. Because of the overdetermined property, the equations cannot be expected to have any solutions if noise perturbs a number of the quantities appearing in the equations. This paper has indicated how to replace the problem of solving equations by one of minimizing a performance index in a consistent way. That is, in the noiseless case, minimization of the performance index should recover the same result as solving the equations, and in a low noise case, the result should be close to the noiseless result, with the perturbation continuous in the noise magnitude.

An important problem, linked but separate from the one treated in this paper, is how (numerically) to solve the minimization problem. The corresponding problem in the noiseless case is how to perform localization. For a localization problem to be solvable in polynomial time, it is generally necessary that some special structure holds for the graph; for example, in the case of trilateration graphs, localization can be done in linear time with suitable anchors [1]. We would expect, although we have no formal proof, that such geometries will also be important in ensuring that a noisy localization problem is computationally tractable. The issue in noisy localization will be to obtain a suitable initial iterate, i.e., one that is close to the global minimum of the index and from which the global minimum can be reached through a standard sort of descent algorithm. It is easy to envisage for a trilateration graph, for example, how one could systematically use noisy measurements to construct an initial iterate for use in the index of this paper. As with noiseless localization in a trilateration graph, initial iterates for the position of each nonanchor node would be successively determined.

There are many other localization problems than those relying on just range, using, for example, bearings and time differences of arrival (TDOA) [17, 16] in which, again in the noiseless case, an overdetermined set of equations determines the solution. For example, in TDOA localization in two dimensions, typically three or more hyperbola branches have a common point of intersection. The same issue will arise in the presence of noise, and this paper gives some of the formal machinery for dealing with it.

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