Capacity of Large Wireless Networks with Generally Distributed Nodes

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***Abstract*—This paper investigates the capacity of a random network in which the nodes have a general spatial distribution. Our model assumes n nodes in a unit square, with a pair of nodes directly connected if and only if their Euclidean distance is smaller than or equal to a threshold, known as the transmission range. Each link has an identical capacity of W bits/s. The transmission range is the same for all nodes and can be any value so long as the resulting network is connected. A capacity upper bound is obtained for the above network, which is valid for both finite n and asymptotically infinite n. We further investigate the capacity upper bound and lower bound for the above network as $n \to \infty$ and show that both bounds can be expressed as a product of four factors, which represents respectively the impact of node distribution, link capacity, number of source destination pairs and the transmission range. The bounds are tight in that the upper bound and lower bound differ by a constant multiplicative factor only. For the special case of networks with nodes distributed uniformly or following a homogeneous Poisson distribution, the bounds are of the same order as known results in the literature.**

***Index Terms*—Capacity, general node distribution, wireless networks.**

I. Introduction

SINCE the seminal work of Gupta and Kumar [1], extensive research has been done on studying the capacity of large wireless networks under various scenarios [1]–[9]. More specifically, in [1], Gupta and Kumar considered an ad-hoc network with a total of n nodes uniformly and *i.i.d.* on an area of unit size. Each node in the network is capable of transmitting at W bits/s and using a fixed and identical transmission range. It was shown that when each node randomly and independently chooses another node in the network as its destination, the transport capacity and the achievable per-node throughput are $\Theta_n\left(W\sqrt{\frac{n}{\log n}}\right)$ and $\Theta_n\left(\frac{W}{\sqrt{n\log n}}\right)$ respectively[1]. When the nodes are optimally and *deterministically* placed to maximize throughput, the transport capacity and the achievable per-node throughput become $\Theta_n\left(W\sqrt{n}\right)$ and $\Theta_n\left(\frac{W}{\sqrt{n}}\right)$ respectively. In [2], Franceschetti *et al.* considered essentially the same random network as that in [1] except that nodes are now allowed to use two different transmission ranges. The link capacity is determined by the associated SINR through the Shannon–Hartley theorem. By having each source-destination pair transmitting via the so-called "highway system", formed by nodes using the smaller transmission range, it was shown in [2] that the transport capacity and the per-node throughput can also reach $\Theta_n\left(\sqrt{n}\right)$ and $\Theta_n\left(\frac{1}{\sqrt{n}}\right)$ respectively even when nodes are randomly deployed. The existence of such highways was demonstrated analytically using the continuum percolation theory [10]. The key to achieving a higher capacity in the network considered in [2] is that nodes are restricted to use the smaller transmission range as often as possible and the larger transmission range can only be used by source (destination) nodes to access their respective nearest highway nodes. In this way, the number of concurrent transmissions that can be accommodated in the network area is maximized, hence the improvement in capacity. In [4] Grossglauser and Tse showed that in mobile ad hoc networks, by leveraging on the nodes' mobility, a per-node throughput of $\Theta_n\left(1\right)$ can be achieved at the expense of a large delay. Their work [4] has sparked tremendous interest in studying the capacity-delay tradeoffs in mobile networks assuming various mobility models and the obtained results often vary greatly with the different mobility models being considered, see [3], [5], [11]–[14] and references therein for examples. In [7], Chen et al. studied the capacity of wireless networks under a different traffic distribution. More specifically, they considered a network with a set of n randomly deployed nodes transmitting to single sink or multiple sinks where the sinks can be either regularly-deployed or randomly-deployed. Under the above settings, it was shown that with single sink, the transport capacity is given by $\Theta_n\left(W\right)$; with k sinks, the transport capacity is increased to $\Theta_n\left(kW\right)$ when $k = O_n(n\log n)$ or $\Theta_n\left(n\log nW\right)$ when $k = \Omega_n\left(n\log n\right)$. In a more recent work [15], Chen et al. further studied the transport capacity of a network with n arbitrarily distributed

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[1]The following notations are used throughout the paper. For two positive functions $f(x)$ and $h(x)$:

- $f(x) = o_x(h(x))$ iff (if and only if) $\lim_{x\to\infty}\frac{f(x)}{h(x)} = 0$ or $\lim_{x\to 0}\frac{f(x)}{h(x)} = 0$;
- $f(x) = \omega_x(h(x))$ iff $h(x) = o_x(f(x))$;
- $f(x) = \Theta_x(h(x))$ iff there exist a sufficiently large x_0 and two positive constants c_1 and c_2 such that for any $x > x_0$, $c_1 h(x) \geq f(x) \geq c_2 h(x)$;
- $f(x) \sim_x h(x)$ iff $\lim_{x\to\infty}\frac{f(x)}{h(x)} = 1$ or $\lim_{x\to 0}\frac{f(x)}{h(x)} = 1$;
- $f(x) = O_x(h(x))$ iff there exist a sufficiently large x_0 and a positive constant c such that for any $x > x_0$, $f(x) \leq ch(x)$;
- $f(x) = \Omega_x(h(x))$ iff $h(x) = O_x(f(x))$;
- An event ξ_x depending on x is said to occur asymptotically almost surely (a.a.s.) if its probability tends to one as $x \to \infty$.

nodes and single sink. In [16], Zhang et al. studied the impact of directional antennas on the transport capacity of a network with n randomly deployed nodes and single sink. In [17], Ji et al. studied the transport capacity of continuous data collection in dual-radio multi-channel networks with single sink and proposed a multi-path scheduling algorithm for continuous data collection in these networks. There is also a significant amount of work studying the impact of infrastructure nodes [6] and multiple-access protocols [9] on capacity and the multicast capacity [8], [18]. We refer readers to [19] for a comprehensive review of related work.

Almost all of the existing work studying the (asymptotic) capacity of random networks assumes that nodes in the network are either uniformly distributed or distributed following homogeneous Poisson distributions. While uniform and Poisson distributions form an important class of spatial distributions and have been extensively used in the area, their capability in capturing the spatial distribution of users in various scenarios and application settings is limited. Therefore it is critical to investigate to what extent the above results on network capacity depend on the underlying node distribution being uniform or Poisson. It is worth noting that in a series of papers [20]–[22], Alfano and Garetto *et al.* studied the capacity of a class of clustered networks in which nodes are distributed according to a doubly stochastic shot-noise Cox process. [A doubly stochastic shot-noise Cox process is formed by first deploying a set of nodes, termed cluster centers, randomly and independently in the network area, and then each cluster center generates independently a point process of nodes with a designated density function.] Consequently, the overall node process is given by the superposition of the individual processes generated by the cluster centers. Their work [20]–[22] generated interesting insight on the performance impact of the doubly stochastic shot-noise Cox process, compared with commonly used Poisson or uniform spatial node distribution. Different from the work of Alfano and Garetto et al. [20]–[22], in this paper we study the capacity of random networks under a more general node distribution, to be defined in Section II.

In addition to capacity, delay is also an important performance metric that has been extensively investigated. In this paper we focus on the study of capacity. We refer readers to [3], [5], [11]–[13] for relevant work on delay and to [23], [24] for relevant work on network connectivity.

The following is a detailed summary of our contributions:

- We develop a novel method of analyzing information exchanges across a closed curve for studying the capacity of large wireless networks, distinct from the methods used in work following the methodology in [1];
- Using the method, we derive the capacity upper bound of networks with generally distributed nodes. The capacity upper bound is valid for both finite and asymptotic infinite networks. The method is shown to be effective and efficient for analyzing the capacity of large networks in a more general setting;
- We derive a necessary condition on the transmission range required for networks with generally distributed nodes to be asymptotically almost surely (a.a.s.) connected;
- Using the above necessary condition on the transmission range, we simplify the capacity upper bound to gain insight on the interactions among major capacity-impacting factors. We show that the asymptotic capacity upper bound for networks with generally distributed nodes is determined by four factors, i.e. node distribution, the link capacity, the number of source-destination pairs and transmission range, in a multiplicative form. The tightness of the upper bound is validated by comparing the upper bound with known results obtained assuming uniform or Poisson node distribution;
- We analyze the lower bound of the asymptotic capacity of networks with generally distributed nodes and show that the lower bound can also be expressed as a product of the four factors, i.e. node distribution, the link capacity, the number of source-destination pairs and transmission range. The method used to obtain the lower bound has been used in other papers. However in order to present the lower bound as the product of the four factors, some novel results need to be established using the continuum percolation theory. These set our work apart from existing results in the literature.

Despite the intellectual challenges in the analysis, our results on the upper bound and lower bound of the asymptotic capacity of networks with generally distributed nodes are presented in a simple form. This simplicity helps to deliver an intuitive understanding on the impact of the four main factors, i.e. node distribution, the link capacity, the number of source-destination pairs and transmission range, on the asymptotic capacity of networks with general node distribution and the interactions among these four main factors. Further, both the analytical techniques developed in the paper and some intermediate results derived when analyzing the capacity are expected to contribute to the network capacity analysis under more general settings.

The rest of this paper is organized as follows: Section II presents the network model of interest; Section III presents theoretical analysis on the capacity upper bound of networks with generally distributed nodes; in Section IV, we conduct a deep examination of the condition that the network is connected and show that the condition implies that we can impose some mild restrictions on the transmission range; in Section V, using the restrictions on the transmission range, we simplify the results obtained in Section III for asymptotically infinite networks and discuss insight revealed through the simplified results. The tightness of the capacity upper bound is also validated; Section VI presents a lower bound on the capacity of asymptotically infinite networks; finally, Section VII concludes this paper.

II. Network Model

Two network models are widely used in the study of (asymptotic) network capacity: the *dense network model* and the *extended network model*. The dense network model considers that the network is deployed in a finite area with a sufficiently large node density, while the extended network model considers that the node density is fixed and the network area is sufficiently large. By appropriate scaling of the distance, the results obtained under one model can often be readily extended to the other model [25], [26].

In this paper, we consider the dense network model. More specifically, we consider a wireless multihop network with a total of n nodes *i.i.d.* on a unit square $A = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ following a general density function $f(\boldsymbol{x})$ where $f(\boldsymbol{x}) > 0$, $\forall \boldsymbol{x} \in A$ and $\int_A f(\boldsymbol{x})\, d\boldsymbol{x} = 1$. For convenience, we further assume that f is a continuous function. Let

$$c_1 = \min_{\boldsymbol{x}\in A} f(\boldsymbol{x}) \quad \text{and} \quad c_2 = \max_{\boldsymbol{x}\in A} f(\boldsymbol{x}). \tag{1}$$

Further, a pair of nodes are directly connected if and only if (iff) their Euclidean distance is smaller than or equal to $r(n)$, known as the transmission range, and the capacity of each link is W bits/s. Here $r(n)$ can be any value (or any function of n) as long as the resulting network is connected, which is a prerequisite for studying the capacity of the network. For simplicity, we may drop the dependence on n for notational brevity and use r and $r(n)$ interchangeably. The above model, widely known as the disk model, captures the key feature in wireless communications that direct communications often occur between nearby devices at predesignated link capacities. Of course, more sophisticated models have been used in the literature. For example, both the existence of a link and its capacity may be determined by the associated SINR [2]. It is often the case that under certain conditions, the results obtained assuming the disk model can be extended to the more sophisticated models [1], [19]. Even when analytic extension is not possible, the disk results can be powerful predictors of what happens in simulations with more sophisticated models.

We consider a scenario where each node chooses randomly and independently another node in the network as its destination. Therefore there are a total of n source-destination pairs in the network. Further, a saturated traffic scenario is considered where each node always has a packet to transmit when a transmission opportunity becomes available. Each node transmits following a CSMA protocol at a common and fixed transmission power [2]. That is, before transmitting, a node first senses the channel and can only transmit if there is no other active transmitter within $(1+\triangle)\, r(n)$, where $\triangle$ is a positive constant and $(1+\triangle)\, r(n)$ is commonly known as the sensing range. Thus two simultaneously active transmitters are separated by a distance of at least $(1+\triangle)\, r(n)$. Note that in the widely used protocol model or the SINR model [1], a distance can also be identified such that two simultaneously active transmitters must be separated by at least that distance. Therefore the communication model used in this paper can be readily extended to incorporate other models. Given the transmission power, the path loss model and the carrier-sensing threshold, the sensing range can be easily computed. We refer readers to [9] and our previous work [27] for more details of CSMA protocols. Further, a random backoff mechanism, which is commonly adopted in CSMA protocols, is often used to resolve channel contention when multiple nodes contend for transmission. Therefore we consider (as is common) an ideal scenario where there is no packet loss due to collision.

[2]Because in our network model, all nodes use the same transmission power (or range), our results are not directly comparable with those in [2], [9], where nodes are allowed to use different transmission ranges. As explained in Section I, when nodes are allowed to use different transmission ranges, the number of concurrent transmissions is maximized. Hence the capacity can be greatly improved.

Denote the above network by $\mathcal{G}(n, r, A)$. In this work, we are interested in finding the capacity of $\mathcal{G}(n, r, A)$.

In particular, we study the capacity of $\mathcal{G}(n, r, A)$ by investigating the so-called per-node throughout. Let Φ be the set of all spatial and temporal routing and scheduling algorithms. Let $\lambda_i^\chi(n)$ be the *long-term* average throughput obtained by the i^{th} source-destination pair when $\chi \in \Phi$. The per-node throughput of $\mathcal{G}(n, r, A)$ when using $\chi \in \Phi$, denoted by $\lambda^\chi(n)$, is given by $\lambda^\chi(n) = \min_{i\in\Gamma} \lambda_i^\chi(n)$ where Γ is the set of indices of all source-destination pairs. The per-node throughput of $\mathcal{G}(n, r, A)$, denoted by $\lambda(n)$, is given by

$$\lambda(n) = \max_{\chi\in\Phi} \left(\lambda^\chi(n)\right) = \max_{\chi\in\Phi} \left(\min_{i\in\Gamma} \lambda_i^\chi(n)\right). \tag{2}$$

That is, there exists a routing and scheduling algorithm and a sufficiently large time interval τ such that every τ time interval, each and every source can transmit at least $\lambda(n)\,\tau$ bits to its destination simultaneously with all other source-destination pairs in the network. This definition is both natural and consistent with that used in [1].

III. A Capacity Upper Bound for Networks with Generally Distributed Nodes

In this section we analyze the capacity upper bound for networks with generally distributed nodes. The obtained upper bound is valid for both finite and asymptotically infinite random networks.

The set of concurrent transmitters using the CSMA protocol at a particular time instant is commonly modeled by a marked point process, known as a Matern process or the hardcore process [19], [28]. In particular, let $\boldsymbol{x}_i$ be the location of node i. A random number uniformly distributed within $[0, 1]$ is assigned to each node. Denote the number assigned to node i by p_i and it is assumed that p_i and p_j are independent where $i \neq j$. A node i is an active transmitter at the time instant if [28]

$$p_i < \min_{j\in\{k:\|\boldsymbol{x}_k-\boldsymbol{x}_i\|\le(1+\triangle)r\}/\{i\}} p_j. \tag{3}$$

That is, its p_i is the smallest among all nodes within its sensing range.

We will use a disk method to determine the capacity upper bound. Let $D(R)$ be a disk centered at the origin and with a radius R where $0 < R \le \frac{1}{2}$, as illustrated in Fig. 1. For a randomly chosen node, the probability that the node falls within $D(R)$ is given by

$$\eta(R) = \int_{D(R)} f(\boldsymbol{x})\, d\boldsymbol{x} \tag{4}$$

and the probability that the node falls outside $D(R)$ is given by $1 - \eta(R)$. It follows that the expected number of nodes within $D(R)$ is $\eta(R)\, n$. For any node within $D(R)$, with probability $1 - \eta(R)$ its destination is located outside $D(R)$. Therefore the expected fraction of source-destination pairs with the source and the destination located on different sides of the boundary of $D(R)$ is given by $2\eta(R)(1 - \eta(R))$.

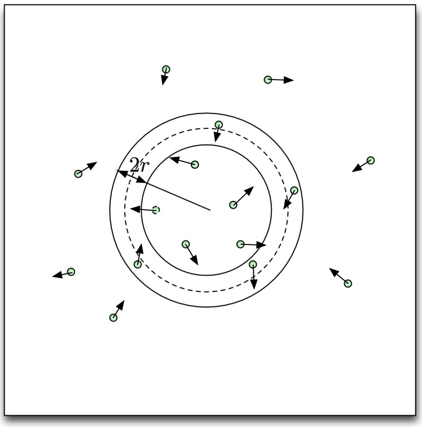

Fig. 1. An illustration of using the disk method to compute the capacity upper bound. The dotted circle represents a circle centered at the origin and with a radius R. The two solid circles represent two circles centered at the origin and with radii $R+r$ and $R-r$ respectively. The arrow represents the direction of packet forwarding.

Denoting by $m(R)$ the expected number of active links crossing the boundary of $D(R)$, the following inequality must hold:

$$2\eta(R)(1-\eta(R))\,n\lambda(n) \le m(R)\,W \tag{5}$$

for any value of R so long as $D(R) \subset A$. As an easy consequence of the above inequality

$$\lambda(n) \le \min_{0<R\le 0.5} \frac{m(R)\,W}{2\eta(R)(1-\eta(R))\,n}. \tag{6}$$

In the following analysis, we will aim to find the value of $m(R)$.

Consider Fig. 1 where the boundary of $D(R)$ is shown as the dotted circle. Let $D(R-r,R+r)$ be an annulus centered at the origin and with an inner radius of $R-r$ and an outer radius of $R+r$. Let $L(R-r,R+r)$ be the random number of simultaneously active transmitters within $D(R-r,R+r)$ in a randomly chosen time slot. If at any time there is an active link crossing the boundary of $D(R)$, there must be an active transmitter within a distance r on either side of the boundary. Hence by averaging we obtain

$$m(R) \le E(L(R-r,R+r)). \tag{7}$$

As with the number of links, the number of simultaneously active transmitters is random and time-varying.

Next we shall find $E(L(R-r,R+r))$. Let dA be a very small (differential) area and $dA \subset D(R-r,R+r)$. Let $\boldsymbol{x} \in dA$ be the center of dA. Denote by η_{dA} the event that there is at least one node in dA. Using the property that nodes are i.i.d. on A following $f(\boldsymbol{x})$, it is evident that

$$\Pr(\eta_{dA}) = 1-(1-f(dA))^n = nf(dA) + o_{dA}(f(dA)). \tag{8}$$

For convenience, we use $f(dA)$ for $\int_{dA} f(\boldsymbol{x})\,d\boldsymbol{x}$. When $dA \to 0$, $f(dA) \to f(\boldsymbol{x})\,dA$.

Let $D(dA,(1+\triangle)r)$ be a disk centered at the center of dA and with a radius $(1+\triangle)r$. The pmf (probability mass function) of the random number of nodes falling into $D(dA,(1+\triangle)r)$, denoted by $N(dA,(1+\triangle)r)$, is given by:

$$\begin{aligned}&\Pr(N(dA,(1+\triangle)r)=k)\\&=\binom{n}{k}[f(D(dA,(1+\triangle)r))]^k\\&\times[1-f(D(dA,(1+\triangle)r))]^{n-k}.\end{aligned}$$

Using the above equation and the expression for $\Pr(\eta_{dA})$ in (8), the joint distribution $\Pr(N(dA,(1+\triangle)r)=k,\eta_{dA})$ can be obtained:

$$\begin{aligned}&\Pr(N(dA,(1+\triangle)r)=k,\eta_{dA})\\&=\Pr(\eta_{dA}|N(dA,(1+\triangle)r)=k)\Pr(N(dA,(1+\triangle)r)=k)\\&=\binom{n}{k}[f(D(dA,(1+\triangle)r))]^{k-1}\\&\times[1-f(D(dA,(1+\triangle)r))]^{n-k}kf(dA)\\&+o_{dA}\left(\frac{f(dA)}{f(D(dA,(1+\triangle)r))}\right).\end{aligned}$$

Conditioned on the two events $N(dA,(1+\triangle)r)=k$ and η_{dA}, denoting by ξ_{dA} the event that there is exactly one node in dA[3] *and* that node is an active transmitter, it can be shown that

$$\begin{aligned}&\Pr(\xi_{dA}|N(dA,(1+\triangle)r)=k,\eta_{dA})\\&=\int_0^1(1-x)^{k-1}dx=\frac{1}{k}\end{aligned} \tag{9}$$

where in the above equation the term $(1-x)^{k-1}$ is the probability that all other $k-1$ nodes in $D(dA,(1+\triangle)r)$, which are competing for transmission opportunities with the node in dA, have their respective values of p larger than x conditioned on that the node in dA has its value of p equal to x. Equation (9) implies that the node in dA has equal opportunity to transmit compared with other nodes in its contention domain.

From the above equations, it follows that

$$\begin{aligned}&\Pr(\xi_{dA})\\&=\Pr(\xi_{dA},\eta_{dA})\\&=\sum_{k=1}^{n}[\Pr(\xi_{dA}|N(dA,(1+\triangle)r)=k,\eta_{dA})\\&\times\Pr(N(dA,(1+\triangle)r)=k,\eta_{dA})]\\&=\sum_{k=1}^{n}\{[f(D(dA,(1+\triangle)r))]^k[1-f(D(dA,(1+\triangle)r))]^{n-k}\\&\times\binom{n}{k}\frac{f(dA)}{f(D(dA,(1+\triangle)r))}+o_{dA}\left(\frac{f(dA)}{f(D(dA,(1+\triangle)r))}\right)\}\\&=\frac{f(dA)}{f(D(dA,(1+\triangle)r))}\sum_{k=1}^{n}\{\binom{n}{k}[f(D(dA,(1+\triangle)r))]^k\\&\times[1-f(D(dA,(1+\triangle)r))]^{n-k}\}+o_{dA}(f(dA))\\&=\frac{1-[1-f(D(dA,(1+\triangle)r))]^n}{f(D(dA,(1+\triangle)r))}f(dA)+o_{dA}(f(dA)).\end{aligned} \tag{10}$$

It follows from the above equation that

$$E(L(R-r,R+r))$$

[3]Following some simple argument, it can be shown that when $dA \to 0$, the probability that there is more than one node in dA is negligible.

$$= \int_{L(R-r,R+r)} \frac{1-\left[1-f\left(D\left(\boldsymbol{x},(1+\triangle)r\right)\right)\right]^n}{f\left(D\left(\boldsymbol{x},(1+\triangle)r\right)\right)} f(\boldsymbol{x})\, dA. \tag{11}$$

Summarizing the four equations (4), (6), (7) and (11), a main result on the capacity upper bound of networks with generally distributed nodes is obtained:

Theorem 1. *For both finite n and asymptotically infinite n, the per-node throughput of $\mathcal{G}(n,r,A)$ satisfies*

$$\lambda(n) \le \min_{0<R\le 0.5} \frac{W \int_{L(R-r,R+r)} \frac{1-[1-f(D(\boldsymbol{x},(1+\triangle)r))]^n}{f(D(\boldsymbol{x},(1+\triangle)r))} f(\boldsymbol{x})\, dA}{2n \int_{D(R)} f(\boldsymbol{x})\, d\boldsymbol{x} \left(1-\int_{D(R)} f(\boldsymbol{x})\, d\boldsymbol{x}\right)}. \tag{12}$$

Remark 1. In the analysis, we used a disk, i.e. $D(R)$, essentially for convenience. In (12), $D(R)$ can be replaced by a connected area of any shape, satisfying the condition that the Minkowski sum [28] of the area and a disk centered at the origin and with radius r is entirely contained in A, and the inequality on the capacity upper bound will remain valid.

IV. Transmission Range Required for a Connected Network

The capacity upper bound obtained in the last section, see (12), is valid for both finite n and therefore asymptotically infinite n, and for both uniformly and Poissonly distributed nodes and nodes distributed under a more general distribution. The analytical form of the capacity upper bound in (12) is however complicated and it is obviously difficult to extract key information on the impact of major capacity-impacting factors. In the rest of this paper, we continue to investigate the *asymptotic* capacity of networks with generally distributed nodes as $n \to \infty$ because it turns out that the above capacity upper bound can be greatly simplified in the asymptotic regime. The simplified result helps to give insight on the main factors that determine the capacity and their interactions. Unless otherwise specified, the capacity bounds obtained in the rest of this paper are only valid as $n \to \infty$.

In this section we establish some conditions on $r(n)$ required for an a.a.s. connected network. These conditions will then be exploited in Sections V and VI to simply the results on capacity.

First, we argue that in the asymptotic regime when $n \to \infty$, it is reasonable to impose the following condition on $r(n)$:

$$r(n) \to 0. \tag{13}$$

Recall that in this paper, a dense network model is used. It is a natural outcome of the dense network model that, as $n \to \infty$, a smaller and smaller transmission range, i.e. $r(n) \to 0$, is used to improve the spatial frequency reuse and hence the capacity; otherwise the entire network may end up being able to have a small and constant number (i.e. non-increasing with n) of simultaneous transmissions only, despite an asymptotically infinite number of nodes competing for channel access. Consequently, the analysis becomes trivial if condition (13) is not met.

Secondly, we shall show that the requirement set out in Section II that the network is connected implies that we can impose a second condition on the transmission range $r(n)$ when analyzing the asymptotic capacity of the network, viz.:

$$\liminf_{n\to\infty} \frac{r(n)}{\sqrt{\frac{\log n + b}{c_1 \pi n}}} \ge 1. \tag{14}$$

Recall that $c_1 = \min_{\boldsymbol{x}\in A} f(\boldsymbol{x})$. Parameter b in (14) is a non-negative real number and satisfies the properties that $b = o_n(\log n)$ (b equals a non-negative constant is allowed).

In the rest of this section, we shall focus on analyzing the sufficient and necessary condition on $r(n)$ required for $\mathcal{G}(n,r,A)$ to have no isolated nodes a.a.s. Note that $\mathcal{G}(n,r,A)$ having no isolated nodes is a necessary condition for $\mathcal{G}(n,r,A)$ to form a connected network. The analysis thus provides justification for condition (14).

The main result of this section is summarized in the following theorem:

Theorem 2. *Let $A_C = \{\boldsymbol{y} \in A : f(\boldsymbol{y}) = c_1\}$, $|A_C|$ be the Lebesgue measure of A_C and*

$$r(n) = \sqrt{\frac{\log n + b}{c_1 \pi n}}. \tag{15}$$

The distribution of the number of isolated nodes in $\mathcal{G}(n,r,A)$ converges to a Poisson distribution with mean $c_1 e^{-b}|A_C|$. Further, with $B = A \backslash A_C$, a.a.s. there is no isolated node in B as $n \to \infty$.

Proof: See Appendix. ■

The following result readily follows from Theorem 2:

Corollary 1. *Under the same settings of Theorem 2, when $|A_C| = 0$, a.a.s. $\mathcal{G}(n,r,A)$ has no isolated node; when $|A_C| > 0$, $\mathcal{G}(n,r,A)$ has no isolated node a.a.s. iff $b \to \infty$ as $n \to \infty$.*

V. A Capacity Upper Bound for Asymptotically Infinite Networks

In this section, by imposing some mild conditions on the transmission range $r(n)$, particularly (13) and (14) which have been justified in Section IV, we simplify the upper bound in (12) for asymptotic infinite networks in a bid to extract the major factors that determine the capacity and understand their roles.

In particular under the two conditions (14) and (13), it can be shown that

$$f\left(D\left(\boldsymbol{x},(1+\triangle)r\right)\right) \sim_n f(\boldsymbol{x})\pi(1+\triangle)^2 r^2$$

$$\text{and} \quad \left[1-f(\boldsymbol{x})\pi(1+\triangle)^2 r^2\right]^n = e^{n\log\left[1-f(\boldsymbol{x})\pi(1+\triangle)^2 r^2\right]} \sim_n e^{-nf(\boldsymbol{x})\pi(1+\triangle)^2 r^2}$$

where in the last step, $\log(1-x) \sim_x -x$ as $x \to 0$ is used. Therefore, $\left[1-f(\boldsymbol{x})\pi(1+\triangle)^2 r^2\right]^n \to 0$ as $n \to \infty$ as a consequence of (14). Using the above two results and (12), it follows that:

$$\lambda(n)$$

$$
\begin{aligned}
&\le \min_{0<R\le 0.5} \frac{W\int_{L(R-r,R+r)} \frac{1-[1-f(D(\boldsymbol{x},(1+\triangle)r))]^n}{f(D(x,(1+\triangle)r))} f(\boldsymbol{x})\,dA}{2n\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left(1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)} \\
&\sim_n \min_{0<R\le 0.5} \frac{\frac{4R}{(1+\triangle)^2 r} W}{2n\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left[1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right]} \\
&= \min_{0<R\le 0.5} \frac{\frac{2R}{(1+\triangle)^2}}{\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left[1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right]} W\frac{1}{n}\frac{1}{r}
\end{aligned} \tag{16}
$$

where the second step results because of the following derivation:

$$
\begin{aligned}
&\int_{L(R-r,R+r)} \frac{1-[1-f(D(\boldsymbol{x},(1+\triangle)r))]^n}{f(D(\boldsymbol{x},(1+\triangle)r))} f(\boldsymbol{x})\,dA \\
\sim_n\ &\int_{L(R-r,R+r)} \frac{1-\left[1-f(\boldsymbol{x})\pi(1+\triangle)^2 r^2\right]^n}{f(\boldsymbol{x})\pi(1+\triangle)^2 r^2} f(\boldsymbol{x})\,dA \\
\sim_n\ &\int_{L(R-r,R+r)} \frac{1}{\pi(1+\triangle)^2 r^2}\,dA = \frac{4R}{(1+\triangle)^2 r}.
\end{aligned}
$$

Let

$$
\beta_f \triangleq \min_{0<R\le 0.5} \frac{\frac{2R}{(1+\triangle)^2}}{\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left[1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right]} \tag{17}
$$

where the subscript f emphasizes the dependence of β on the node distribution function f, (16) can be written in neater form as that given in the following theorem:

Theorem 3. *As $n\to\infty$, asymptotically almost surely the per-node throughput of $\mathcal{G}(n,r,A)$ satisfies*

$$
\lim_{n\to\infty} \lambda(n) \le \lim_{n\to\infty} \beta_f W \times \frac{1}{n} \times \frac{1}{r}. \tag{18}
$$

In (18), the parameter β_f captures the impact of node distribution and *is also entirely determined by the node distribution*. The parameter W represents the impact of link capacity. The parameter $\frac{1}{n}$ represents the impact of the number of source-destination pairs sharing the network capacity. Finally the parameter $\frac{1}{r}$ represents the impact of the transmission range and shows that the network capacity (upper bound) is inversely proportional to the transmission range. Equation (18) shows that the network capacity (upper bound) can be expressed as the product of the above four terms.

Remark 2. In this paper, following the most commonly used model in the field [1], it is assumed that the number of source-destination pairs is equal to the number of nodes in the network. Hence the parameter n has the dual meaning of being both the number of nodes and the number of source-destination pairs. We can also consider a scenario where the number of source-destination pairs is not equal to the number of nodes. Let m be the number of source-destination pairs. Then under some mild conditions, i.e. each node has an equal probability, albeit smaller than 1, to become a traffic source independently and the respective destinations of the source nodes are uniformly and independently chosen from the remaining nodes, the capacity scaling law suggested in (18) also holds with the parameter n being replaced by the parameter m.

In the rest of this section, we validate the tightness of the capacity upper bound by first computing the upper bound for a special case that has been widely investigated, i.e. networks with uniformly distributed nodes, then comparing the obtained upper bound with the known results for networks with this distribution.

For networks with uniformly distributed nodes, $f(\boldsymbol{x})=1$. It follows from (17) that

$$
\begin{aligned}
\beta_f &= \min_{0<R\le 0.5} \frac{\frac{2R}{(1+\triangle)^2}}{\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left[1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right]} \\
&= \min_{0<R\le 0.5} \frac{2}{\pi(1+\triangle)^2} \times \frac{1}{R(1-\pi R^2)} \\
&= \frac{4}{3\pi(1+\triangle)^2\sqrt{3\pi}}.
\end{aligned}
$$

Combining the above equation with (18), an upper bound on the per-node throughput of networks with uniformly distributed nodes results:

$$
\lambda(n) \le \frac{4}{3\pi(1+\triangle)^2\sqrt{3\pi}} W \times \frac{1}{n} \times \frac{1}{r}. \tag{19}
$$

It is well known that the critical (minimum) transmission range required for a network with a total of n nodes uniformly distributed in a unit area to be a.a.s. connected is [24], [29]

$$
r^c(n) = \sqrt{\frac{\log n + c(n)}{\pi n}} \tag{20}
$$

where $c(n) = o_n(\log n)$ and $c(n)\to\infty$ as $n\to\infty$.

As an easy consequence of (19) and (20), as $n\to\infty$

$$
\lambda(n) \le \frac{4}{3\pi(1+\triangle)^2\sqrt{3}} W \times \frac{1}{\sqrt{n(\log n + c(n))}}.
$$

It is well known that the per-node throughput of networks with uniformly distributed node is $\Theta_n\left(\frac{W}{\sqrt{n\log n}}\right)$ [1]. Therefore the capacity upper bound obtained in (18) is tight in the sense that when applied to the special case of networks with uniformly distributed nodes, the capacity upper bound is in the same order of the capacity of networks with uniformly distributed nodes. Due to the close relationship between uniform and Poisson distributions [30], the above conclusion can be readily extended to networks with nodes distributed in a unit square following a homogeneous Poisson distribution with density n.

VI. A Capacity Lower Bound for Asymptotically Infinite Networks

In this section, we derive a capacity lower bound for asymptotically infinite networks. This is done by using a constructive method. More specifically, we shall first construct a deterministic (i.e. the set of active transmitters at a particular time instant is not random but scheduled *a priori*) scheduling and routing algorithm obeying the carrier-sensing constraint of CSMA networks and obtain the minimum capacity that can be achieved using the algorithm. Then using the result in [9, Lemma 9], which states that by adjusting the countdown rates of nodes, where the countdown rate is a controllable parameter in CSMA protocols, a distributed scheduling algorithm exists

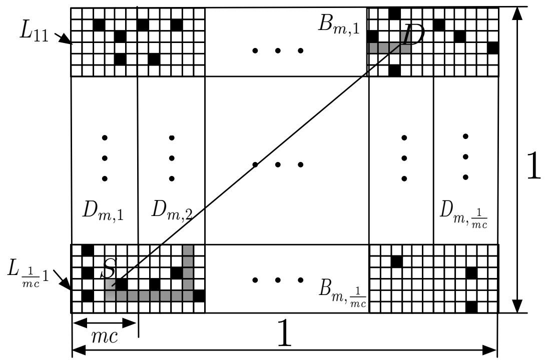

Fig. 2. An illustration of the partitioning of A. Black square represents a closed site and white square represents an open site. Note that $L_{ij} = B_{m,i} \cap D_{m,j}$. Packets between a randomly source S and its destination D will be routed through these open sites entirely contained in L_{ij}s intersecting the line segment connecting S and D, as shown via grey squares in the figure.

such that the CSMA network in which each node accesses the channel randomly following the distributed CSMA protocols can achieve a capacity that is larger than or equal to that achieved by its deterministic counterpart, the capacity lower bound for the CSMA network being studied readily follows.

In the rest of this section, we construct the deterministic routing and scheduling algorithm and study the capacity of the network under the algorithm. Before we explain our routing algorithm, we need to first establish some preliminary results.

We partition A into squares of size c^2 where $c = \frac{r}{\sqrt{5}}$. Following common terminology used in percolation theory, we also refer to these squares as *sites* and use the two terms squares and sites interchangeably. Let $m = \lceil 5\pi \rceil = 16$ where $\lceil 5\pi \rceil$ is the smallest integer larger than 5π. We also partition A into $\frac{1}{mc}$ horizontal slices, where each slice is a rectangle of size $1 \times mc$. Obviously there are $\frac{1}{c} \times m$ squares in each slice. See Fig. 2 for an illustration of the partitioning. Modifications to handle the situation that $\frac{1}{mc}$ may not be an integer are well known [2], [10]. Therefore in the following analysis, we ignore the possibility that $\frac{1}{mc}$ may not be an integer and assume that $\frac{1}{mc}$ is an integer. Denote the i-th rectangle by $B_{m,i}$ where $1 \le i \le \frac{1}{mc}$. We call a site (or a square) an *open* site if there is at least one node in it; call a site a *closed* site if there is no node in it. Let p_j be the probability that the j-th site is an open site and let $A_j \subseteq A$ denote that site. It is straightforward to show that

$$p_j = 1 - (1 - f(A_j))^n \ge 1 - \left(1 - c_1 c^2\right)^n \triangleq p. \qquad (21)$$

The site is closed with probability $1 - p_j \le 1 - p$. Further, the event that a site is closed (or open) and the event that another site is closed (or open) are asymptotically independent as $n \to \infty$ [10], [25].

We describe two sites as *adjacent* sites if they share a common edge. Obviously if two sites are adjacent, any node located in the first site (if exists) and any node located in the second site must be directly connected. We define a *left to right crossing of open sites* in $B_{m,i}$ as a sequence of distinct open sites that starts from an open site on the left border of $B_{m,i}$ and ends at an open site on the right border of $B_{m,i}$ such that all sites are entirely contained in $B_{m,i}$ and sites next to each other in the sequence are adjacent sites. Denote by $B^{\leftrightarrow}_{m,i}$ the event that there exists a left to right crossing of open sites in $B_{m,i}$. The following result on $B^{\leftrightarrow}_{m,i}$ can be established:

Lemma 1. *There is a left to right crossing of open sites in each $B_{m,i}$, $1 \le i \le \frac{1}{mc}$, a.a.s. as $n \to \infty$.*

Proof: Obviously the complement of $B^{\leftrightarrow}_{m,i}$ is the event that there exists a *top to bottom crossing of closed sites* in $B_{m,i}$ [10]. That is, there is a sequence of distinct closed sites in $B_{m,i}$ that starts from the top border of $B_{m,i}$ and ends at the bottom border of $B_{m,i}$ where sites next to each other in the sequence are adjacent sites. Denote by $\overline{B^{\leftrightarrow}_{m,i}}$ the event that there is a top-bottom crossing of closed sites in $B_{m,i}$. It suffices to show that $\lim_{n\to\infty} \Pr\left(\cup_{1 \le i \le \frac{1}{mc}} \overline{B^{\leftrightarrow}_{m,i}}\right) = 0$.

Let Λ denote a randomly chosen site in the top border of $B_{m,i}$. A top to bottom crossing of sites in $B_{m,i}$ (with the open or closed status of each site at this point irrelevant) starting from Λ comprises *at least* m sites. The total number of sequences of distinct sites in $B_{m,i}$ of length m starting from Λ, where sites next to each other in the sequence are adjacent sites, is not larger than 3^{m-1}. This is because starting from Λ, in each step, only one out of (up to) three sites can be chosen to add into the sequence. The probability that all m sites in this sequence of sites are closed sites is upper bounded by $(1-p)^m$. As a consequence of the union bound, the probability that there is a sequence of m closed sites starting from Λ *and* sites next to each other in the sequence are adjacent, is upper bounded by $3^{m-1}(1-p)^m$. Noting that for a sufficiently large n, $3(1-p) < 1$ and hence $3^{k-1}(1-p)^k$ is a decreasing function of k. Therefore, if a top to bottom crossing of closed sites in $B_{m,i}$, starting from Λ, has more than m closed sites, its probability will only be smaller than $3^{m-1}(1-p)^m$.

Using the union bound and noting that the total number of sites in the top border of $B_{m,i}$ is $\frac{1}{c}$, it follows from the above analysis that

$$\Pr\left(\overline{B^{\leftrightarrow}_{m,i}}\right) \le \frac{1}{c} 3^{m-1} (1-p)^m .$$

Using the union bound again and (21), an upper bound on the probability that there exists a top to bottom crossing of closed sites in *any* of the $B_{m,i}$, $1 \le i \le \frac{1}{mc}$, can be obtained:

$$\begin{aligned} \Pr\left(\cup_i \overline{B^{\leftrightarrow}_{m,i}}\right) &\le \frac{1}{mc} \times \frac{1}{c} 3^{m-1} (1-p)^m \\ &= \frac{1}{mc^2} 3^{m-1} \left(1 - c_1 c^2\right)^{nm} . \end{aligned}$$

Noting that $m = \lceil 5\pi \rceil$ and using $c = \frac{r}{\sqrt{5}}$ and the lower bound on r given in (14) to derive the first inequality, it can be further established that

$$\begin{aligned} &\lim_{n\to\infty} \Pr\left(\cup_i \overline{B^{\leftrightarrow}_{m,i}}\right) \\ \le\ & \lim_{n\to\infty} \frac{5 c_1 \pi n}{m(\log n + b)} 3^{m-1} \left(1 - \frac{\log n + b}{5\pi n}\right)^{nm} \\ =\ & \lim_{n\to\infty} \frac{5 c_1 \pi n}{m(\log n + b)} 3^{m-1} e^{nm \log\left(1 - \frac{\log n + b}{5\pi n}\right)} = 0. \end{aligned}$$

■

The implication of Lemma 1 is that there is a path starting from a node on the left border of $B_{m,i}$ and ending at a node on the right border of $B_{m,i}$ in each $B_{m,i}$, $1 \leq i \leq \frac{1}{mc}$. Further, each path does not "wiggle" more than a mc amount, i.e. each path cannot have a difference in the maximum and minimum distances from the bottom of the square by more than mc.

By symmetry, if we partition A into $\frac{1}{mc}$ vertical slices, denoted by $D_{m,j}$ where $1 \leq j \leq \frac{1}{mc}$ (see Fig. 2 for an illustration) and each slice is a rectangle of size $mc \times 1$, it follows that a.a.s. there is a top to bottom crossing of open sites in each $D_{m,j}$, $1 \leq j \leq \frac{1}{mc}$.

Based on Lemma 1 and the above result on the existence of a top to bottom crossing of open sites in $D_{m,j}$, $1 \leq j \leq \frac{1}{mc}$, it can be shown that an arbitrarily chosen node in the non-boundary area of A, if it is not already located in the site that forms one of the left-right or top-bottom crossings of open sites, is enclosed in a region bounded by two left-right crossings of open sites and two top-bottom crossings of open sites. Further the two left-right (top-bottom) crossings of open sites are separated by an Euclidean distance of at most $2mc$. Now using the property that the network is connected and that nodes are directly connected to each other following a disk model, it readily follows that there exists a (multi-hop) path between the node and a node located in a site that forms one of the four crossings and the two nodes are separated by an Euclidean distance of no more than $2\sqrt{2}mc$. A similar argument can also be easily made for nodes located near the four sides of A or located at the four corners of A.

We are now ready to introduce our routing algorithm. Let $L_{ij} = B_{m,i} \cap D_{m,j}$, $1 \leq i, j \leq \frac{1}{mc}$. It can be readily shown that a.a.s. every L_{ij} contains both a left to right crossing of open sites and a top to bottom crossing of open sites. After having reached a node located in the site that forms a (left-right or top-bottom) crossing of open sites, using some simple geometric arguments, it can be established that if we route the packet in a zigzag pattern along these left-right and top-bottom crossings of open sites contained in L_{ij}s intersecting the straight line segment connecting the source and the destination, the path deviates from the line segment by no more than $\sqrt{2}mc$. See Fig. 2 for an illustration of the routing algorithm. The above discussions are summarized in the following lemma:

Lemma 2. *For every source-destination pair in* A*, as* $n \to \infty$*, a.a.s. there is path connecting the source and the destination and the path deviates from the line segment connecting the source and the destination by no more than* $2\sqrt{2}mc$*.*

Lemma 2 implies that packets between every source-destination pair can be routed along a path which may have many turns, but never deviates far in distance from a straight line connecting the respective source-destination pairs. If there are more than one paths that deviate from the source-destination line by no more than $2\sqrt{2}mc$, a path is chosen randomly among the possibilities to route the packets between the source and the destination.

Next we analyze the maximum number of source-destination paths passing an arbitrarily chosen site in order to obtain an upper bound on the traffic load of the site.

Let $\boldsymbol{y}_i \in A_i$ be the center of an arbitrarily chosen site A_i. Consider a randomly chosen source node S located at distance x from $\boldsymbol{y}_i$. Let $C(\boldsymbol{y}_i)$ be the disk centered at $\boldsymbol{y}_i$ with a radius

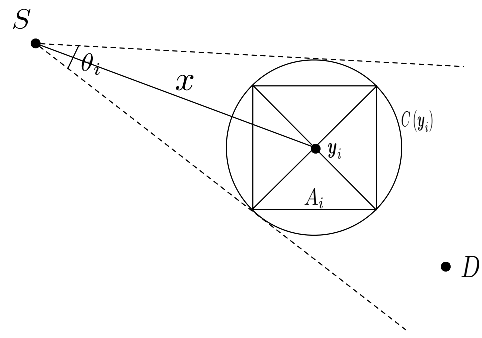

Fig. 3. An illustration of the analysis of the traffic load of a site.

$2\sqrt{2}mc$ and let θ_i be the angle subtended by $C(\boldsymbol{y}_i)$ at S. See 3 for an illustration. It can be shown that $\theta_i = 2\arcsin\frac{2\sqrt{2}mc}{x}$.

Since all nodes are located on a unit square A, the size of the area bounded by the two dashed lines and the border of A in Fig. 3 is at most $\frac{\theta_i}{2\pi}\pi\left(\sqrt{2}\right)^2 = \theta_i$. Denote the area by A_{θ_i}. Denote by ς_S the event that the destination of S is located in A_{θ_i}. It can be shown:

$$\Pr(\varsigma_S) = \int_{A_{\theta_i}} f(x)\,dx \leq c_2\theta_i. \tag{22}$$

(Recall that $c_2 = \max_{\boldsymbol{x}\in A} f(\boldsymbol{x})$.) Let $h(y)$ be the probability density function that a randomly chosen node is located at distance y from $\boldsymbol{y}_i$. Denote by $D(\boldsymbol{y}_i, x, x+\triangle x)$ an annulus centered at $\boldsymbol{y}_i$ and with an inner radius of x and an outer radius $x+\triangle x$. First it can be shown that

$$h(x) = \lim_{\triangle x\to 0}\frac{\int_{D(\boldsymbol{y}_i,x,x+\triangle x)\cap A} f(\boldsymbol{x})\,d\boldsymbol{x}}{\triangle x} \leq c_2 2\pi x. \tag{23}$$

Define an indicator random variable Y_{ij} such that $Y_{ij}=1$ if a randomly chosen source-destination line (say, labelled as the j-th source-destination pair) passes through $C(\boldsymbol{y}_i)$; $Y_{ij}=0$ otherwise. Let $N_i = \sum_{j=1}^{n} Y_{ij}$. Using $\theta_i = 2\arcsin\frac{2\sqrt{2}mc}{x}$, (22), (23) and further considering the situation that if S is located in $C(\boldsymbol{y}_i)$ (a randomly chosen node has probability $\pi\left(2\sqrt{2}mc\right)^2 = 8\pi m^2c^2$ to be in $C(\boldsymbol{y}_i)$), the source-destination line must passes $C(\boldsymbol{y}_i)$, it can be shown that

$$\begin{aligned}
&E(N_i)\\
\leq& n\left(\int_{2\sqrt{2}mc}^{\sqrt{2}} \Pr(\varsigma_S)\,h(x)\,dx + 8\pi m^2c^2\right)\\
\leq& n\left(\int_{2\sqrt{2}mc}^{\sqrt{2}} \left(2\arcsin\frac{2\sqrt{2}mc}{x}\right) c_2^2 2\pi x dx + 8\pi m^2c^2\right)\\
\leq& n\left(c_2^2 16\sqrt{2}\pi mc + 8\pi m^2c^2\right)
\end{aligned}$$

where in the last step $\arcsin x \leq 2x$ is used.

Let $\beta = c_2^2 16\sqrt{2}\pi mc + 8\pi m^2c^2$ for convenience and define a new set of i.i.d. indicator random variable Z_j, $1 \leq j \leq n$ such that $\Pr(Z_j = 1) = \beta$ and $Z = \sum_{j=1}^{n} Z_j$. Our further analysis needs to use the following ordering result. For two real-valued random variables X_1 and X_2, we say $X_1 \leq_{st} X_2$ iff for all $x \in (-\infty, \infty)$, $\Pr(X_1 > x) \leq \Pr(X_2 > x)$.

Theorem 4. *[31, Theorem 1(a)]Suppose X_i follows a Binomial distribution with parameters $n_i \in \mathbb{N}$ and $p_i \in (0,1)$, denote the distribution of X_i by $B(n_i, p_i)$, $i = 1, 2$, i.e. $X_i \sim B(n_i, p_i)$. We have $X_1 \leq_{st} X_2$ iff $(1-p_1)^{n_1} \geq (1-p_2)^{n_2}$ and $n_1 \leq n_2$.*

As an easy consequence of the above theorem, for three independent Binomial random variables $X_1 \sim B(n_1, p_1)$, $X_2 \sim B(n_1, p_2)$ and $X_3 \sim B(n_2, p_2)$ with $n_1 \leq n_2$ and $p_1 \leq p_2$, it can be concluded that $X_1 \leq_{st} X_2 \leq_{st} X_3$.

Using the above result, it follows that $N_i \leq_{st} Z$. Therefore,

$$\begin{aligned}&\Pr(N_i \geq (1+\delta)n\beta) \leq \Pr(Z \geq (1+\delta)n\beta)\\ &= \Pr(Z \geq (1+\delta)E(Z)) \leq e^{-\frac{\delta^2 E(Z)}{3}} = e^{-\frac{\delta^2}{3}n\beta}\end{aligned}$$

where δ is a small positive constant and in the third step, the Chernoff bound is used.

On the basis of the above result, and using the union bound and (14)

$$\begin{aligned}&\lim_{n\to\infty} \Pr(\cup_i (N_i \geq (1+\delta)n\beta))\\ &\leq \lim_{n\to\infty} \frac{1}{c^2} e^{-\frac{\delta^2}{3}n\beta}\\ &= \lim_{n\to\infty} \{e^{-\left[\frac{n\delta^2}{3}\left(c_2^2 16\sqrt{2}\pi m\sqrt{\frac{\log n+b}{5c_1\pi n}} + 8\pi m^2 \frac{\log n+b}{5c_1\pi n}\right)\right]}\\ &\times \frac{5c_1\pi n}{\log n + b}\} = 0 \end{aligned} \tag{24}$$

Using Lemma 2, a node located in a site will only carry the traffic of a source-destination pair if the associated source-destination line segment intersects the disk centered at the site and with a radius $2\sqrt{2}mc$. Therefore as an easy consequence of (24), a.a.s the number of source-destination lines passing through an arbitrarily chosen site, including the source-destination lines originating from or ending at the site, is bounded by $(1+\delta)n\beta$ for every site in A.

Further it can be easily shown that there exists a scheduling algorithm and a positive *constant* $C_\triangle$, depending on $\triangle$, such that an open site can become active (i.e. a node in the site is transmitting) for at least $\frac{1}{C_\triangle}$ fraction of time.

It follows that

$$\begin{aligned}&\lim_{n\to\infty} \lambda(n)\\ \geq\ &\lim_{n\to\infty} \frac{W}{C_\triangle} \times \frac{1}{(1+\delta)n\beta}\\ =\ &\lim_{n\to\infty} \frac{W}{C_\triangle} \times \frac{1}{(1+\delta)n\left[c_2^2 256\sqrt{\frac{2}{5}}\pi r + \frac{2048\pi}{5}r^2\right]}\\ =\ &\lim_{n\to\infty} \frac{1}{C_\triangle(1+\delta)c_2^2 256\sqrt{\frac{2}{5}}\pi} \times W \times \frac{1}{n} \times \frac{1}{r}\end{aligned} \tag{25}$$

where (13) is used in the last step. The above result is summarized in the following theorem:

Theorem 5. *As $n \to \infty$, asymptotically almost surely the per-node throughput of $\mathcal{G}(n, r, A)$ satisfies*

$$\lim_{n\to\infty} \lambda(n) \geq \lim_{n\to\infty} \frac{1}{C_\triangle(1+\delta)c_2^2 256\sqrt{\frac{2}{5}}\pi} \times W \times \frac{1}{n} \times \frac{1}{r} \tag{26}$$

Equation (26) reveals that the lower bound of the capacity of asymptotically infinite networks with a general node distribution can also be expressed as the product of four factors: parameter $\frac{1}{C_\triangle(1+\delta)c_2^2 256\sqrt{\frac{2}{5}}\pi}$ captures the impact of node distribution and is also entirely determined by the node distribution. Parameter W represents the impact of link capacity. Parameter $\frac{1}{n}$ represents the impact of the number of source-destination pairs sharing the network capacity. Finally parameter $\frac{1}{r}$ represents the impact of the transmission range and shows that the network capacity (lower bound) is inversely proportional to the transmission range. Further, the lower bound in (26) and the upper bound in (18) varies by a constant multiplicative factor only.

Remark 3. As manifested in (26), the capacity lower bound is solely determined by the maximum value of $f(\boldsymbol{x})$. This suggests that the traffic bottleneck in a network is in the areas with denser nodes. When more knowledge on $f(\boldsymbol{x})$ is available, it is possible to design more intelligent routing algorithms to bypass these traffic hot spots, thereby reducing the gap between the capacity upper bound and the capacity lower bound. Further, other things being equal, uniform distribution is the optimum distribution among the class of random distributions that maximizes the capacity lower bound in (26).

VII. Conclusion

In this paper, we studied the capacity of networks with generally distributed nodes. More specifically, we considered random networks with a total of n nodes i.i.d. on a unit square following a general distribution function. Further, a pair of nodes are directly connected following a unit disk model with a transmission range $r(n)$ and two simultaneously active transmitters have to be separated by at least $(1+\triangle)r(n)$ due to the use of carrier sensing. The transmission range $r(n)$ is the same for all nodes and can take any value as long as the resulting network is connected. A capacity upper bound of the above network is obtained by analyzing the number of links crossing a simple closed curve. The capacity upper bound is valid for both finite networks and asymptotic infinite networks.

To simplify the analytical expression and to gain insight, we conducted a deep examination of the condition that the network is connected and showed that the condition implies that we can impose some mild restrictions on the transmission range. Utilizing these constraints on $r(n)$, both the asymptotic capacity upper bound and the asymptotic capacity lower bound have been shown to be the product of four terms: a term determined by the spatial node distribution only, the link capacity W, $\frac{1}{n}$ where n represents the number of source-destination pairs sharing the network capacity, and $\frac{1}{r(n)}$ which represents the combined impact of spatial frequency reuse and the number of relays required for end-to-end deliver information. The results suggest that the impact of the spatial node distribution on the network capacity can be captured by a single parameter and the spatial node distribution only affect the network capacity by up to a constant multiplicative factor. The upper and lower bounds are tight in the sense that they differ by a constant multiplicative factor only and for the special case of networks with uniformly distributed nodes, the bounds are in the same order of the capacity of networks with uniformly distributed nodes.

We expect that both the analytical techniques developed in the paper and some intermediate results derived when analyzing the capacity, e.g. an expression for the transmission range required for an a.a.s. connected network with generally distributed nodes, and a demonstration that packets between every source-destination pair can be routed along almost straight lines connecting the respective source-destination pairs, to contribute to the network capacity analysis under more general settings.

Our result also reveals that the traffic bottleneck in the network is likely to be in the areas with denser nodes. When more knowledge on $f(\boldsymbol{x})$ is available, it may be possible to design more intelligent routing algorithms to bypass these traffic hot spots, thereby reducing the gap between the capacity upper bound and the capacity lower bound.

Appendix

In this Appendix, we prove Theorem 2.

Denoted by ξ the random number of isolated nodes in A, we first analyze the expected value of ξ. Then we analyze the asymptotic distribution of the number of isolated nodes in a network with the same node distribution and transmission range as $\mathcal{G}(n,r,A)$ however with nodes distributed on a unit torus. Denote the latter network on a torus by $\mathcal{G}^T\left(n,r,A^T\right)$ and denote the random number of isolated nodes in $\mathcal{G}^T\left(n,r,A^T\right)$ by ξ^T. We show that the distribution of ξ^T converges to a Poisson distribution with mean $c_1e^{-b}|A_C|$. Using the property that $\xi^T\le_{st}\xi$, the property that $E\left(\xi^T\right)=E(\xi)$ (both properties to be proved later), together with the property that both random variables are non-negative integers, we are able to conclude that as $n\to\infty$, ξ^T and ξ converge to the same distribution. Thus, Theorem 2 readily follows. For two real-valued random variables X_1 and X_2, we say $X_1\le_{st}X_2$ iff for all $x\in(-\infty,\infty)$, $\Pr(X_1>x)\le\Pr(X_2>x)$.

Considering a differential area $dA\subset A$, the probability that there is exactly one node in dA is given by $nf(dA)(1-f(dA))^{n-1}=nf(dA)+o_{dA}(f(dA))$ and the probability that there is more than one node in dA is negligible compared with $nf(dA)$. Let $\boldsymbol{x}$ be the center of dA. Denote by η_{dA} the event that there is exactly one node in dA and denote by ξ_{dA} the event that there is one node in dA *and* the node is isolated. It can be shown that

$$\begin{aligned}&\Pr(\xi_{dA})=\Pr(\xi_{dA}|\eta_{dA})\Pr(\eta_{dA})\\&=[1-f(D(\boldsymbol{x},r)\setminus dA)]^{n-1}nf(dA)(1-f(dA))^{n-1}\\&+o_{dA}(f(dA))\end{aligned}\tag{27}$$

where $D(\boldsymbol{x},r)$ denotes a disk centered at $\boldsymbol{x}$ and with a radius r. It follows from the above equation that:

$$\begin{aligned}E(\xi)&=\int_A[1-f(D(\boldsymbol{x},r)\setminus dA)]^{n-1}nf(dA)\\&=\int_A e^{(n-1)\log[1-f(D(\boldsymbol{x},r))]}nf(\boldsymbol{x})\,dA\\&\sim_n\int_A e^{-(n-1)f(\boldsymbol{x})\pi r^2}nf(\boldsymbol{x})\,dA\end{aligned}$$

where in the last step, $r(n)=\sqrt{\frac{\log n+b}{c_1\pi n}}$, $f(D(\boldsymbol{x},r))\sim_n f(\boldsymbol{x})\pi r^2$ as $n\to\infty$ and $\log(1-x)\sim_x -x$ as $x\to 0$ are used. Recall that $f(x)\in[c_1,c_2]$, it follows from the above equation that

$$\begin{aligned}\lim_{n\to\infty}E(\xi)&=\lim_{n\to\infty}\int_A e^{-(n-1)f(\boldsymbol{x})\frac{\log n+b}{c_1 n}}nf(\boldsymbol{x})\,dA\\&=\lim_{n\to\infty}\int_{A_C}e^{-b}f(\boldsymbol{x})\,dA=c_1e^{-b}|A_C|\end{aligned}\tag{28}$$

Now let us consider another network with the same node distribution and transmission range as $\mathcal{G}(n,r,A)$ however with nodes distributed on a unit torus $A^T=\left[-\frac{1}{2},\frac{1}{2}\right]^2$. Denote the network on the unit torus by $\mathcal{G}^T\left(n,r,A^T\right)$.

The use of a toroidal rather than planar region as a tool in analyzing network properties is well known [32]. The unit torus $A^T=[-\frac{1}{2},\frac{1}{2}]^2$ that is commonly used in random geometric graph theory is essentially the same as a unit square $A=[-\frac{1}{2},\frac{1}{2}]^2$ except that the distance between two points on a torus is defined by their *toroidal distance*, instead of Euclidean distance. Thus a pair of nodes in $\mathcal{G}^T\left(n,r,A^T\right)$, located at $\boldsymbol{x}_1\in A^T$ and $\boldsymbol{x}_2\in A^T$ respectively, are directly connected iff their *toroidal distance*, denoted by $\|\boldsymbol{x}_1-\boldsymbol{x}_2\|^T$, is smaller than or equal to $r(n)$. For a unit torus $A^T=[-\frac{1}{2},\frac{1}{2}]^2$, the toroidal distance is given by $\|\boldsymbol{x}_1-\boldsymbol{x}_2\|^T\triangleq\min\{\|\boldsymbol{x}_1+\boldsymbol{z}-\boldsymbol{x}_2\|:\boldsymbol{z}\in\mathbb{Z}^2\}$ [32, p. 13].

We note the following relation between toroidal distance and Euclidean distance on a square area centered at the origin:

$$\|\boldsymbol{x}_1-\boldsymbol{x}_2\|^T\le\|\boldsymbol{x}_1-\boldsymbol{x}_2\|\quad\text{and}\quad\|\boldsymbol{x}\|^T=\|\boldsymbol{x}\|\tag{29}$$

which will be used in the later analysis.

Denote by ξ^T the random number of isolated nodes in $\mathcal{G}^T\left(n,r,A^T\right)$. Now we use the coupling technique [10] to construct the connection between ξ and ξ^T. Note that given a random instance of $\mathcal{G}^T\left(n,r,A^T\right)$, a random instance of $\mathcal{G}(n,r,A)$ can be obtained by removing connections between nodes in $\mathcal{G}^T\left(n,r,A^T\right)$ whose toroidal distance is smaller than or equal to $r(n)$ but whose Euclidean distance is larger than $r(n)$, and the converse. Since the removal of connections will increase or keep the number of isolated nodes in the network. Therefore

$$\xi^T\le_{st}\xi.\tag{30}$$

It readily follows that a necessary condition for $\mathcal{G}(n,r,A)$ to have no isolated nodes a.a.s. is that $\mathcal{G}^T\left(n,r,A^T\right)$ has no isolated nodes a.a.s.

Next we shall show that $r(n)=\sqrt{\frac{\log n+b}{c_1\pi n}}$ is a sufficient and necessary condition for $\mathcal{G}^T\left(n,r,A^T\right)$ to have no isolated node a.a.s. Our further analysis relies on the use of the Chen-Stein bound [30], [33]. We first establish some preliminary results that allow us to use the Chen-Stein bound for the analysis of number of isolated nodes in $\mathcal{G}^T\left(n,r,A^T\right)$.

Divide the network area A^T into m^2 non-overlapping squares each with size $\frac{1}{m^2}$. Denote the i_m^{th} square by A_{i_m}. Define two sets of indicator random variables J_{i_m} and I_{i_m} with $i_m\in\Gamma_m\triangleq\{1,\dots m^2\}$, where $J_{i_m}=1$ iff there exists exactly one node in A_{i_m}, otherwise $J_{i_m}=0$; $I_{i_m}=1$ iff there is exactly one node in A_{i_m} *and* that node is isolated, $I_{i_m}=0$ otherwise. Obviously J_{i_m} is independent of $J_{j_m},j_m\in\Gamma_m\setminus\{i_m\}$. Denote the center of A_{i_m} by $\boldsymbol{x}_{i_m}$ and without loss of generality we assume that when $J_{i_m} = 1$, the associated node in A_{i_m} is at $\boldsymbol{x}_{i_m}$[4]. Observe that for any fixed m, the values of $\Pr(I_{i_m} = 1)$ and $\Pr(J_{i_m} = 1)$ do not depend on the particular index i_m on a torus. However both the set of indices Γ_m and a particular index i_m depend on m. As m changes, the square associated with $I_{i_m}^T$ and $J_{i_m}^T$ also changes. In this paper, we are only interested in the limiting values of the above parameters associated with a sub-square as $m \to \infty$. Without causing any confusion, we drop the subscript m for convenience in the following analysis.

First it can be shown that

$$\begin{aligned}\Pr(J_i = 1) &= n\int_{A_i} f(\boldsymbol{x})\,d\boldsymbol{x}\left(1 - \int_{A_i} f(\boldsymbol{x})\,d\boldsymbol{x}\right)^{n-1} \\ &\sim_m \frac{n}{m^2} f(\boldsymbol{x}_i)\,. \end{aligned} \tag{31}$$

Let us consider $\Pr(I_i = 1 | J_i = 1)$ now, i.e the probability that given exactly one node in A_i, the node is isolated. Let $D^T(\boldsymbol{x}_i, r)$ be a disk of radius r centered at $\boldsymbol{x}_i$ in the torus, i.e. $D^T(\boldsymbol{x}_i, r) = \left\{\boldsymbol{y} \in A^T : \|\boldsymbol{y} - \boldsymbol{x}_i\|^T \le r\right\}$. It can be readily shown that

$$\begin{aligned}&\Pr(I_i = 1 | J_i = 1) \\ \sim_m &\left(1 - \int_{D^T(\boldsymbol{x}_i, r)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)^{n-1}. \end{aligned} \tag{32}$$

Combining the two equations (31) and (32), it follows that

$$\begin{aligned}&\Pr(I_i = 1) \\ \sim_m &\frac{n}{m^2} f(\boldsymbol{x}_i)\left(1 - \int_{D^T(\boldsymbol{x}_i, r)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)^{n-1}. \end{aligned} \tag{33}$$

Now consider the event $I_i I_j = 1, i \ne j$, conditioned on the event that $J_i J_j = 1$, meaning that both nodes having been placed inside A_i and A_j respectively are isolated. It can be shown that

$$\begin{aligned}&\Pr(I_i I_j = 1 | J_i J_j = 1) \\ =&1\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T > r\right)\left(1 - \int_{D(\boldsymbol{x}_i, r) \cup D(\boldsymbol{x}_j, r)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)^{n-2}. \end{aligned} \tag{34}$$

where $1\left(\left\|\boldsymbol{x}_i - \boldsymbol{x}_j^T\right\| > r\right)$ is an indicator function and $1\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T > r\right) = 1$ iff $\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T > r$; $1\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T > r\right) = 0$ otherwise. The inclusion of the term $1\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T > r\right)$ is due to the requirement that the two nodes located inside A_i and A_j respectively cannot be directly connected given that they are both isolated nodes. Further, it can be shown that

$$\begin{aligned}&\Pr(J_i J_j = 1) \\ \sim_m &n(n-1) f(\boldsymbol{x}_i) f(\boldsymbol{x}_j)\left(\frac{1}{m^2}\right)^2. \end{aligned} \tag{35}$$

[4]In this paper we are mainly concerned with the case that $m \to \infty$, i.e. the size of the square is vanishingly small. Therefore the actual position of the node in the square is not important.

Combining the two equations (34) and (35):

$$\begin{aligned}&\Pr(I_i I_j = 1) \\ &\sim_m n(n-1) f(\boldsymbol{x}_i) f(\boldsymbol{x}_j)\left(\frac{1}{m^2}\right)^2 1\left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T > r\right) \\ &\times\left(1 - \int_{D^T(\boldsymbol{x}_i, r) \cup D^T(\boldsymbol{x}_j, r)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)^{n-2}. \end{aligned} \tag{36}$$

Noting that when $\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T \ge 2r$,

$$\begin{aligned}&\int_{D^T(\boldsymbol{x}_i, r) \cup D^T(\boldsymbol{x}_j, r)} f(\boldsymbol{x})\,d\boldsymbol{x} \\ =&\int_{D^T(\boldsymbol{x}_i, r)} f(\boldsymbol{x})\,d\boldsymbol{x} + \int_{D(\boldsymbol{x}_j, r)} f(\boldsymbol{x})\,d\boldsymbol{x}. \end{aligned}$$

It follows that when $\|\boldsymbol{x}_i - \boldsymbol{x}_j\|^T \ge 2r$,

$$\begin{aligned}&\lim_{m\to\infty} \frac{\Pr(I_i I_j = 1)}{\Pr(I_i = 1)\Pr(I_j = 1)} \\ &\sim_n \frac{\left(1 - \int_{D^T(\boldsymbol{x}_i, r)} f(\boldsymbol{x})\,d\boldsymbol{x} - \int_{D^T(\boldsymbol{x}_j, r)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)^{n-2}}{(1 - \int_{D^T(\boldsymbol{x}_i, r)} f(\boldsymbol{x})\,d\boldsymbol{x})^{n-1}(1 - \int_{D^T(\boldsymbol{x}_j, r)} f(\boldsymbol{x})\,d\boldsymbol{x})^{n-1}} \\ &\sim_n \exp\{-(n-2)(\int_{D^T(\boldsymbol{x}_i, r)} f(\boldsymbol{x})\,dx + \int_{D^T(\boldsymbol{x}_j, r)} f(\boldsymbol{x})\,d\boldsymbol{x}) \\ &\quad + (n-1)\int_{D^T(\boldsymbol{x}_i, r)} f(\boldsymbol{x})\,dx + (n-1)\int_{D^T(\boldsymbol{x}_i, r)} f(\boldsymbol{x})\,d\boldsymbol{x}\} \\ &\sim_n e^{\int_{D^T(\boldsymbol{x}_i, r)} f(\boldsymbol{x})dx + \int_{D^T(\boldsymbol{x}_j, r)} f(\boldsymbol{x})d\boldsymbol{x}} \sim_n 1. \end{aligned} \tag{37}$$

where in the second step, $\log(1-x) \sim_x -x$ as $x \to 0$ is used. Equation (37) implies that the event that a node is isolated and the event that another node separated by a distance of more than $2r$ from the first node is also isolated are asymptotically independent as $n \to \infty$.

As an easy consequence of (33), it can be shown that

$$\begin{aligned}&E\left(\xi^T\right) \\ &= \lim_{m\to\infty} E\left(\sum_{i \in \Gamma_m} I_i\right) \\ &= \lim_{m\to\infty} \sum_{i \in \Gamma_m} \frac{n}{m^2} f(\boldsymbol{x}_i)\left(1 - \int_{D^T(\boldsymbol{x}_i, r)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)^{n-1} \\ &= \int_{A^T} n f(\boldsymbol{y})\left(1 - \int_{D^T(\boldsymbol{y}, r)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)^{n-1} d\boldsymbol{y}. \end{aligned} \tag{38}$$

Based on the above analysis, the following theorem on the asymptotic convergence of $E\left(\xi^T\right)$ as $n \to \infty$ can be proved:

Theorem 6. *Let ξ^T be the number of isolated nodes in $\mathcal{G}^T\left(n, r, A^T\right)$. As $n \to \infty$, $E\left(\xi^T\right)$ converges to a non-negative constant $c_1 e^{-b}|A_C|$ where $c_1 = \min_{\boldsymbol{y} \in A} f(\boldsymbol{y})$, $A_C = \left\{\boldsymbol{y} \in A^T : f(\boldsymbol{y}) = c_1\right\}$ and $|A_C|$ is the Lebesgue measure of A_C. Further, let $B = A^T \backslash A_C$, a.a.s. there is no isolated node in B as $n \to \infty$.*

Proof: It follows from (38) that

$$\lim_{n\to\infty} E\left(\xi^T\right)$$

$$= \lim_{n\to\infty}\int_{A^T} nf(\boldsymbol{y})\left(1-\int_{D^T(\boldsymbol{y},r)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)^{n-1} d\boldsymbol{y}$$
$$= \lim_{n\to\infty}\int_{A^T} f(\boldsymbol{y})\, e^{\log n+(n-1)\log\left(1-f(\boldsymbol{y})\frac{\log n+b}{c_1 n}\right)} dy$$

where (15) is used in the last step. It can be further shown that

$$\begin{aligned}
&\lim_{n\to\infty}\log n+(n-1)\log\left(1-f(\boldsymbol{y})\frac{\log n+b}{c_1 n}\right)\\
&= \lim_{n\to\infty}\frac{\frac{\log n}{n-1}+\log\left(1-f(\boldsymbol{y})\frac{\log n+b}{c_1 n}\right)}{\frac{1}{(n-1)}}\\
&= \lim_{n\to\infty}\frac{\frac{1-\log n}{(n-1)^2}+\frac{-1}{1-f(\boldsymbol{y})\frac{\log n+b}{c_1 n}}\times f(\boldsymbol{y})\frac{-\log n-b+1}{c_1 n^2}}{-(n-1)^{-2}}\\
&= \lim_{n\to\infty}\left(1-\frac{f(\boldsymbol{y})}{c_1}\right)\log n - b\frac{f(\boldsymbol{y})}{c_1}+\left(\frac{f(\boldsymbol{y})}{c_1}-1\right)
\end{aligned}$$

where L'Hï¿½pital's rule is used in the second step of the above derivations. Combining the above two equations

$$\begin{aligned}
&\lim_{n\to\infty} E\left(\xi^T\right)\\
&=\lim_{n\to\infty}\int_{A^T} f(\boldsymbol{y})\, n^{1-\frac{f(\boldsymbol{y})}{c_1}} e^{-b\frac{f(\boldsymbol{y})}{c_1}+\left(\frac{f(\boldsymbol{y})}{c_1}-1\right)} d\boldsymbol{y}\\
&=\int_{A_C} c_1 e^{-b} d\boldsymbol{y}\\
&+\lim_{n\to\infty}\int_{B} f(\boldsymbol{y})\, n^{1-\frac{f(\boldsymbol{y})}{c_1}} e^{-b\frac{f(\boldsymbol{y})}{c_1}+\left(\frac{f(\boldsymbol{y})}{c_1}-1\right)} d\boldsymbol{y} = c_1 e^{-b}\left|A_C\right|.
\end{aligned} \tag{39}$$

Let W_B be the number of isolated nodes in B. As an easy consequence of the above analysis,

$$\begin{aligned}
&\lim_{n\to\infty} E\left(W_B\right)\\
&=\lim_{n\to\infty}\int_{B} f(\boldsymbol{y})\, n^{1-\frac{f(\boldsymbol{y})}{c_1}} e^{-b\frac{f(\boldsymbol{y})}{c_1}+\left(\frac{f(\boldsymbol{y})}{c_1}-1\right)} d\boldsymbol{y} = 0.
\end{aligned}$$

Noting that W_B is a non-negative random integer, the conclusion readily follows that $\lim_{n\to\infty}\Pr\left(W_B=0\right)=1$. ∎

Now we are ready to use the Chen-Stein bound to obtain the asymptotic distribution of the number of isolated nodes in A^T. Below we give a formal statement of the Chen-Stein bound for completeness:

Theorem 7. *[30, Theorem 1.A] For a set of indicator random variables $I_i,\ i\in\Gamma$, define $W\triangleq\sum_{i\in\Gamma} I_i$, $p_i\triangleq E\left(I_i\right)$ and $\eta\triangleq E\left(W\right)$. For any choice of the index set $\Gamma_{s,i}\subset\Gamma$, $\Gamma_{s,i}\cap\{i\}=$ ï¿½,*

$$\begin{aligned}
& d_{TV}(\mathcal{L}(W),Po(\eta))\\
\le\ & \sum_{i\in\Gamma}[(p_i^2+p_iE(\sum_{j\in\Gamma_{s,i}} I_j))]\min(1,\frac{1}{\eta})\\
+\ & \sum_{i\in\Gamma}E(I_i\sum_{j\in\Gamma_{s,i}} I_j)\min(1,\frac{1}{\eta})\\
+\ & \sum_{i\in\Gamma}E|E\{I_i|(I_j,j\in\Gamma_{w,i})\}-p_i|\min(1,\frac{1}{\eta})
\end{aligned}$$

where $\mathcal{L}(W)$ denotes the distribution of W, $Po(\eta)$ denotes a Poisson distribution with mean η, $\Gamma_{w,i}=\Gamma\backslash\left\{\Gamma_{s,i}\cup\{i\}\right\}$ and d_{TV} denotes the total variation distance. The total variation distance between two probability distributions α and β on $\mathbb{Z}^+$ is given by $d_{TV}\left(\alpha,\beta\right)\triangleq\sup\left\{\left|\alpha\left(A\right)-\beta\left(A\right)\right|: A\subset\mathbb{Z}^+\right\}$.

For convenience, we separate the bound in Theorem 7 into three terms $b_1\min(1,\frac{1}{\eta})$, $b_2\min(1,\frac{1}{\eta})$ and $b_3\min(1,\frac{1}{\eta})$ where $b_1\triangleq\sum_{i\in\Gamma}[(p_i^2+p_iE(\sum_{j\in\Gamma_{s,i}} I_j))]$, $b_2\triangleq\sum_{i\in\Gamma}E(I_i\sum_{j\in\Gamma_{s,i}} I_j)$ and $b_3\triangleq\sum_{i\in\Gamma}E|E\{I_i|(I_j,j\in\Gamma_{w,i})\}-p_i|$.

Using the Chen-Stein bound, the following theorem, which provides a stronger result on the distribution of ξ^T than Theorem 6, can be established:

Theorem 8. *Let ξ^T be the number of isolated nodes in $\mathcal{G}^T\left(n,r,A^T\right)$. As $n\to\infty$, the distribution of ξ^T converges to a Poisson distribution with mean $E\left(\xi^T\right)=c_1e^{-b}\left|A_C\right|$.*

Proof: Note that in Theorem 7, $p_i=E(I_i)$ and $E(I_i)=\Pr(I_i=1)$ has been given in (33). Parameter ξ^T is related to parameter W in Theorem 7 by that in each and every random instance of $\mathcal{G}^T\left(n,r,A^T\right)$, the number of isolated nodes is equal to the corresponding value of W as $m\to\infty$. Therefore, $\lim_{m\to\infty}W\overset{d}{=}\xi^T$, where $\overset{d}{=}$ means equal in distribution. The above equation implies that to prove the theorem, it suffice to show that as $n\to\infty$, the distribution of $\lim_{m\to\infty}W$ converges to a Poisson distribution with mean $E\left(\xi^T\right)=c_1e^{-b}\left|A_C\right|$. This result is proved using the Chen-Stein bound.

Let $D^T\left(\boldsymbol{x}_i,r\right)=\{\boldsymbol{x}\in A^T:\left\|\boldsymbol{x}-\boldsymbol{x}_i\right\|^T\le r\}$. Further define the neighborhood of an index $i\in\Gamma$ as $\Gamma_{s,i}\triangleq\{j:\boldsymbol{x}_j\in D\left(\boldsymbol{x}_i,2r\right)\}\backslash\{i\}$ and define the non-neighborhood of the index i as $\Gamma_{w,i}\triangleq\{j:\boldsymbol{x}_j\notin D(\boldsymbol{x}_i,2r)\}$. It can be shown that

$$|\Gamma_{s,i}|=m^2 4\pi r^2+o_m(m^2 4\pi r^2). \tag{40}$$

Further, as an easy consequence Theorem 6, particularly equation (39), $\lim_{n\to\infty}\lim_{m\to\infty}\eta=c_1e^{-b}\left|A_C\right|$, and that $p_i=E(I_i)$, the conclusion follows:

$$\begin{aligned}
&\lim_{m\to\infty}\sum_{i\in\Gamma}p_i\\
&=\int_{A^T} nf(\boldsymbol{y})\left(1-\int_{D^T(\boldsymbol{y},r)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)^{n-1} d\boldsymbol{y},
\end{aligned} \tag{41}$$

$$\lim_{n\to\infty}\lim_{m\to\infty}\sum_{i\in\Gamma}p_i = c_1e^{-b}\left|A_C\right|. \tag{42}$$

Next we shall evaluate the b_1, b_2 and b_3 terms in the following three subsections separately and show that all three terms converge to 0 as $n\to\infty$.

A. An Evaluation of the b_1 Term

Using (33) and (41), it can be shown that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty} b_1\\
&=\lim_{n\to\infty}\lim_{m\to\infty}\sum_{i\in\Gamma}p_iE(\sum_{j\in\Gamma_{s,i}\cup\{i\}} I_j)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\int_{A^T}\{nf(\boldsymbol{y})(1-\int_{D^T(\boldsymbol{y},r)} f(\boldsymbol{x})\,d\boldsymbol{x})^{n-1}\\
&\times[\int_{D^T(\boldsymbol{y},2r)} nf(\boldsymbol{z})(1-\int_{D^T(\boldsymbol{z},r)} f(\boldsymbol{x})\,d\boldsymbol{x})^{n-1}d\boldsymbol{z}]\}d\boldsymbol{y} \quad (43)\\
&= \lim_{n\to\infty}\int_{A^T}\{nf(\boldsymbol{y})(1-\int_{D^T(\boldsymbol{y},r)} f(\boldsymbol{x})\,d\boldsymbol{x})^{n-1}\\
&\times[\int_{D^T(\boldsymbol{y},2r)} f(\boldsymbol{z})\,n^{1-\frac{f(\boldsymbol{z})}{c_1}}e^{-b\frac{f(\boldsymbol{z})}{c_1}+\left(\frac{f(\boldsymbol{z})}{c_1}-1\right)}d\boldsymbol{z}]\}d\boldsymbol{y} \quad (44)\\
&\le \lim_{n\to\infty}\int_{A^T}\{nf(\boldsymbol{y})(1-\int_{D^T(\boldsymbol{y},r)} f(\boldsymbol{x})\,d\boldsymbol{x})^{n-1} \quad (45)\\
&\times[c_2 e^{-b+\left(\frac{f(\boldsymbol{y})}{c_1}-1\right)}4\pi r^2]\}d\boldsymbol{y} \quad (46)\\
&\le c_2 e^{-b+\left(\frac{c_2}{c_1}-1\right)}\lim_{n\to\infty}\{4\pi r^2\int_{A^T} nf(\boldsymbol{y})\\
&\times(1-\int_{D^T(\boldsymbol{y},r)} f(\boldsymbol{x})\,d\boldsymbol{x})^{n-1}d\boldsymbol{y}\} \quad (47)\\
&= c_2 e^{-b+\left(\frac{c_2}{c_1}-1\right)}\lim_{n\to\infty}4\pi r^2 c_1 e^{-b}\,|A_C| = 0 \quad (48)
\end{aligned}
$$

where (33) and (38) are used in obtaining (43); (44) is obtained using the same steps resulting in (39); (45) is obtained by noting that $c_1 = \min_{\boldsymbol{x}\in A} f(\boldsymbol{x})$, $c_2 = \max_{\boldsymbol{x}\in A} f(\boldsymbol{x})$ and that $D^T(\boldsymbol{y},2r)$ asymptotically converges to a single point $\boldsymbol{y}$ as $n\to\infty$; (48) is obtained using Theorem 6, more specifically (39).

B. An Evaluation of the b_2 Term

Now let us consider the b_2 term. It can be shown that

$$
\begin{aligned}
&\lim_{m\to\infty} b_2\\
&= \lim_{m\to\infty}\sum_{i\in\Gamma}(\sum_{j\in\Gamma_{s,i}} E(I_iI_j))\\
&= \lim_{m\to\infty}\sum_{i\in\Gamma}[\sum_{j\in\Gamma_{s,i}} n(n-1)f(\boldsymbol{x}_i)f(\boldsymbol{x}_j)\left(\frac{1}{m^2}\right)^2\\
&\times 1(\|\boldsymbol{x}_i-\boldsymbol{x}_j\|^T > r)(1-\int_{D^T(\boldsymbol{x}_i,r)\cup D^T(\boldsymbol{x}_j,r)} f(\boldsymbol{x})\,d\boldsymbol{x})^{n-2}] \quad (49)\\
&= \int_{A^T}[\int_{D^T(\boldsymbol{y},2r)} n(n-1)f(\boldsymbol{y})f(\boldsymbol{z})1(\|\boldsymbol{y}-\boldsymbol{z}\|^T > r)\\
&\times(1-\int_{D^T(\boldsymbol{y},r)\cup D(\boldsymbol{z},r)} f(\boldsymbol{x})\,d\boldsymbol{x})^{n-2}d\boldsymbol{z}]d\boldsymbol{y} \quad (50)\\
&= \int_{A^T}[\int_{D^T(\boldsymbol{y},r,2r)} n(n-1)f(\boldsymbol{y})f(\boldsymbol{z})\\
&\times(1-\int_{D^T(\boldsymbol{y},r)\cup D^T(\boldsymbol{z},r)} f(\boldsymbol{x})\,d\boldsymbol{x})^{n-2}d\boldsymbol{z}]d\boldsymbol{y}\\
&\sim_n \int_{A^T}[\int_{D^T(\boldsymbol{y},r,2r)} n(n-1)f^2(\boldsymbol{y})\\
&\times\left(1-f(\boldsymbol{y})\left|D^T(\boldsymbol{y},r)\cup D^T(\boldsymbol{z},r)\right|\right)^{n-2}d\boldsymbol{z}]d\boldsymbol{y} \quad (51)
\end{aligned}
$$

where (36) is used in obtaining (49) and $D^T(\boldsymbol{y},r,2r) = D^T(\boldsymbol{y},2r)\setminus D^T(\boldsymbol{y},r)$. Using some straightforward geometric analysis, it can be shown that when $z\in D(y,r,2r)$,

$$
\begin{aligned}
&|D(y,r)\cup D(z,r)|\\
&=2\pi r^2 - 2r^2\arcsin(\sqrt{1-\|y-z\|^2/(4r^2)})\\
&+r\|y-z\|\sqrt{1-\|y-z\|^2/(4r^2)} \ge \frac{4}{3}\pi r^2 + \frac{\sqrt{3}}{2}r^2.
\end{aligned}
$$

Using the above inequality, it follows from (51) that

$$
\begin{aligned}
&\int_{A^T}[\int_{D^T(\boldsymbol{y},r,2r)} n(n-1)f^2(\boldsymbol{y})\\
\times\quad &(1-f(\boldsymbol{y})|D(\boldsymbol{y},r)\cup D(\boldsymbol{z},r)|)^{n-2}d\boldsymbol{z}]d\boldsymbol{y}\\
\le\quad &\int_{A^T}[\int_{D^T(\boldsymbol{y},r,2r)} n(n-1)f^2(\boldsymbol{y})\\
\times\quad &\left(1-f(\boldsymbol{y})(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})r^2\right)^{n-2}d\boldsymbol{z}]d\boldsymbol{y}\\
\sim_n\quad &\int_{A^T}[n(n-1)f^2(\boldsymbol{y})\\
\times\quad &\left(1-f(y)(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})r^2\right)^{n-2}3\pi r^2]d\boldsymbol{y}. \quad (52)
\end{aligned}
$$

Consider the term $n(n-1)\left(1-f(\boldsymbol{y})(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})r^2\right)^{n-2}\pi r^2$ inside the above integral. It can be shown that

$$
\begin{aligned}
&\lim_{n\to\infty} n(n-1)\left(1-f(\boldsymbol{y})(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})r^2\right)^{n-2}\pi r^2\\
&= \lim_{n\to\infty}(n-1)(1-f(\boldsymbol{y})(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})\frac{\log n+b}{c_1\pi n})^{n-2}\frac{\log n+b}{c_1}\\
&\le \lim_{n\to\infty}(n-1)(1-(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})\frac{\log n+b}{\pi n})^{n-2}\frac{\log n+b}{c_1}\\
&=\frac{1}{c_1}\lim_{n\to\infty} e^{\log(n-1)+(n-2)\log\left(1-(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})\frac{\log n+b}{\pi n}\right)+\log(\log n+b)} \quad (53)\\
&=\frac{1}{c_1}\lim_{n\to\infty} e^{\log(n-1)-(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})\frac{(n-2)(\log n+b)}{\pi n}+\log(\log n+b)} = 0 \quad (54)
\end{aligned}
$$

where the equality that $\log(1-x)\sim_x -x$ as $x\to 0$ is used in obtaining (53).

Combining the above equations (50), (51), (52) and (54), conclusion follows that $\lim_{n\to\infty}\lim_{m\to\infty} b_2 = 0$.

C. An Evaluation of the b_3 Term

Using (37), it can be shown that when $j\in\Gamma_{w,i}$

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\frac{\Pr(I_i=1|I_j=1)}{\Pr(I_i=1)}\\
&= \lim_{n\to\infty}\lim_{m\to\infty}\frac{\Pr(I_iI_j=1)}{\Pr(I_i=1)\Pr(I_j=1)} = 1. \quad (55)
\end{aligned}
$$

Using the above equation and noting that $\lim_{n\to\infty}\lim_{m\to\infty}\Pr(I_j=0)=1$, it can be further shown that

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\frac{\Pr(I_i=1|I_j=0)}{\Pr(I_i=1)}\\
&= \lim_{n\to\infty}\lim_{m\to\infty}\frac{\Pr(I_i=1,I_j=0)}{\Pr(I_i=1)\Pr(I_j=0)}\\
&= \lim_{n\to\infty}\lim_{m\to\infty}\frac{\Pr(I_i=1)(1-\Pr(I_j=1))}{\Pr(I_i=1)\Pr(I_j=0)} = 1. \quad (56)
\end{aligned}
$$

Using equations (55), (56), and equation (42), which shows that $\sum_{i\in\Gamma} E(I_i)$ converges to a finite value as $m\to\infty$ and $n\to\infty$, it follows that

$$
\lim_{n\to\infty}\lim_{m\to\infty} b_3
$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i\in\Gamma} E\left|E\left(I_i\right) - p_i\right| = 0.$$

A combination of the analysis in subsections A, B and C completes this proof. ■

Theorem 8, equations (28) and (30), together with the property that both random variables are non-negative integers [26, Lemma 2], allows us to obtain Theorem 2.

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