Analytical Bounds on the Critical Density for Percolation in Wireless Multi-hop Networks

Seh Chun Ng*‡, Guoqiang Mao*‡, and Brian D.O. Anderson†‡
*School of Electrical and Information Engineering, University of Sydney
†Research School of Information Sciences and Engineering, Australian National University
‡National ICT Australia (NICTA), Australia
Email: seh.ng@sydney.edu.au, guoqiang.mao@sydney.edu.au, brian.anderson@anu.edu.au

***Abstract*—In this paper we develop analytical bounds on the critical density for percolation in wireless multi-hop networks, but in contrast to other studies, under a random connection model and with nodes Poissonly distributed in the plane $\mathbb{R}^2$. The establishment of a direct connection between any two nodes follows a random connection model satisfying some intuitively reasonable conditions, i.e. rotational and translational invariance, non-increasing monotonicity and integral boundedness. It is well known that under the above network model and connection model there exists a critical density below which almost surely a fixed but arbitrary node is connected (via single or multi-hop path) to finite number of other nodes only, and above which the node is connected to an infinite number of other nodes with a positive probability. In this paper we investigate the bounds on the critical density. The result is compared with the existing results under a specific connection model, i.e. the unit disk communication model, and it is shown that our method generates bounds close to the known ones. The result provides valuable insight into the design of large-scale wireless multi-hop networks.**

***Index Terms*—random geometric graph, critical density, Poisson random connection model, percolation.**

I. Introduction

In a wireless multi-hop network there are a number of self-organized and decentralized wireless nodes which communicate with each other in a peer-to-peer manner over wireless channels. Each node helps in forwarding packets from source nodes to destination nodes in a multi-hop manner. In such a network, one node must establish a communication path to other nodes in order to successfully transmit and receive information. Therefore, connectivity of the wireless multi-hop network is important for accomplishing many network tasks.

A widely studied problem in wireless multi-hop networks is the node density required to ensure that an arbitrarily chosen node is connected (either directly or via multi-hop path) to infinitely many other nodes with a positive probability. This is the well-known critical density problem in continuum percolation [1], [2], [3] and in this paper we study the connectivity of wireless multi-hop network from percolation perspective. Specifically consider a 2D wireless multi-hop network with nodes distributed following a homogeneous Poisson point process in $\mathbb{R}^2$ with known density λ. The connections between nodes are modeled using a random connection model with the connection function $g : \mathbb{R}^2 \rightarrow [0, 1]$. In the model, two nodes located at $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{y} \in \mathbb{R}^2$ respectively are directly connected with probability $g(\mathbf{x} - \mathbf{y})$, independent of other pairs of nodes. The connection function g satisfies the following properties of rotational and translational invariance, non-increasing monotonicity and integral boundedness [1], [3]:

$$\begin{cases} g(\mathbf{x}) = g(\mathbf{y}) & \text{whenever } \|\mathbf{x}\| = \|\mathbf{y}\|, \\ g(\mathbf{x}) \leq g(\mathbf{y}) & \text{whenever } \|\mathbf{x}\| \geq \|\mathbf{y}\|, \\ 0 < \int_{\mathbb{R}^2} g(\mathbf{x}) d\mathbf{x} < \infty \end{cases} \tag{1}$$

where $\|.\|$ is the Euclidean norm. It is shown that in such network there exists a critical density above which an arbitrarily chosen node is connected to an infinite number of other nodes via a (multi-hop) path with a positive probability, and below which the node is almost surely connected to finite number of other nodes only [1]. We will give a more rigorous definition of the critical density in Section III.

It is beneficial for any wireless multi-hop network to be deployed with node density above the critical density so that each node can communicate with a large number of other nodes in the network. However, the exact value of critical density is hard to obtain. In this paper, we obtain a lower bound on the critical density using a Galton-Watson branching process [3] and an upper bound by coupling the problem to site percolation in a triangular lattice [4]. As a special case of our results we then obtain the bounds under the unit disk communication model, which is a special case of the random connection model considered in our paper. In the unit disk model, two nodes are directly connected if and only if their Euclidean distance is less than or equal to the transmission range r. That is

$$g(\mathbf{x}) = \begin{cases} 1 & \text{if } \|\mathbf{x}\| \leq r, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

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for $\mathbf{x} \in \mathbb{R}^2$. Then we compare our results under the unit disk model with the existing results in the literature. It is shown that our method generates bounds close to the known ones in the literature.

The rest of this paper is organized as follows. In Section II we introduce related work on the critical node density. In Section III we obtain analytically the bounds on the critical density under the random connection model. In Section IV we focus on the unit disk model as a special case and compare our results with other existing results in the literature. Finally conclusions and future work are given in Section V.

II. Related Work

Up to this point the exact value of the critical density is still unknown. Some bounds on the critical density have been given in the literature, but almost exclusively for the unit disk communication model, which is a special case of the random connection model considered in this paper.

Since the results in the literature under the unit disk communication model were obtained with a different transmission range, we rescale the results to a common transmission range of 2, and report them as follows. Balister et al. [5] obtained by *simulation* that the critical density should lie between 0.3587 and 0.3593 with 99.99% confidence. A well-known analytical bound on the critical density was given by Meester and Roy [1], i.e. the critical density should lie between 0.174 and 0.843. The same analytical upper bound but a tighter lower bound (0.2554) were reported by Gu and Hong [6]. Using an analytical technique based on probabilistic methods and the clustering effect in random geometric graph, Kong and Yeh [7] yielded another lower bound (0.1925) on the critical density. No upper bound was obtained in [7].

Results under the unit disk communication model however are often too idealistic for practical use. This is because in the unit disk model, the probability that two nodes are directly connected is a binary function of their Euclidean distance. In reality, the direct connection between a pair of nodes is affected by many other factors, e.g. shadow fading or multi-path fading. The unit disk model cannot capture these complicated factors. Hence, obtaining the results under a more realistic connection model is of paramount importance. In [8], Kong and Yeh further extended the unit disk model to the unit disk model with unreliable links, i.e. two nodes are no longer directly connected with probability 1 but some lesser probability, provided their Euclidean distance is within the transmission range r. The analytical result obtained is comparable with their earlier result under the unit disk model [7]. In [9], Li and Yang obtained analytically an upper bound under the log-normal shadowing model where the upper bound is expressed in the form of a function of the critical density under the unit disk model. Note that both the unit disk model with unreliable links and the log-normal shadowing model are special cases of the random connection model considered in this paper. To the best of our knowledge, there is no result reported for the random connection model.

III. Analysis

In the sequel, we use the Poisson random connection graph to model the wireless multi-hop networks and the connections between nodes. The definition of the Poisson random connection graph [3] is as follows.

Definition 1. Let $\mathcal{H}_\lambda$ denotes a homogeneous Poisson process of density λ on $\mathbb{R}^2$. Let $\mathcal{H}_{\lambda,\mathbf{x_0}}$ denote the point process $\mathcal{H}_\lambda \cup \{\mathbf{x_0}\}$ where $\mathbf{x_0} \in \mathbb{R}^2$. Then the Poisson random connection graph $G(\mathcal{H}_{\lambda,\mathbf{x_0}}; g)$ is a random graph with vertex set $\mathcal{H}_{\lambda,\mathbf{x_0}}$. Any two vertices (with coordinates $\mathbf{x}$ and $\mathbf{y}$ respectively) in the graph are directly connected with probability $g(\mathbf{x}-\mathbf{y})$, independent of other pairs of vertices, where the function g satisfies properties in (1).

In this paper, we use vertex and node interchangeably. Next we define the percolation probability [3].

Definition 2. Let $\mathcal{W}$ be the set of nodes in $G(\mathcal{H}_{\lambda,\mathbf{x_0}}; g)$ connected to the node at $\mathbf{x_0}$ via a multi-hop path. Denote by $|\mathcal{W}|$ the number of nodes in $\mathcal{W}$. Then the percolation probability $\theta(\lambda) = \Pr^\lambda(|\mathcal{W}| = \infty)$ is the probability that $\mathcal{W}$ contains an infinite number of nodes.

Evidently, $\theta(\lambda) > 0$ means that an arbitrarily chosen node is connected to an infinite number of nodes with a positive probability. In addition, it can be shown that $\theta(\lambda)$ is a non-decreasing function of λ and displays the following phase transition phenomenon.

Theorem 1 (Theorem 2.5.1 in [3])**.** *For $G(\mathcal{H}_{\lambda,\mathbf{x_0}}; g)$, there exists a critical density $0 < \lambda_c < \infty$ such that $\theta(\lambda) = 0$ for $\lambda < \lambda_c$ and $\theta(\lambda) > 0$ for $\lambda > \lambda_c$.*

In the following two sub-sections, we obtain analytically an upper bound and a lower bound on λ_c.

A. Lower Bound on λ_c

The following lemma is used in obtaining the lower bound on λ_c.

Lemma 1. *Consider $G(\mathcal{H}_{\lambda,\mathbf{x_0}}; g)$ and denote by X_0 the node at $\mathbf{x_0}$. A node $Y \in \mathcal{H}_\lambda$ is called a k-hop node if the length of the shortest path between Y and X_0, measured by the number of hops, is k. Let N_k be the total number of k-hop nodes. Then*

$$E[N_1] = \int_{\mathbb{R}^2} \lambda g(\mathbf{y}-\mathbf{x_0}) d\mathbf{y} \tag{3}$$

and

$$E[N_k] \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} h_k(\mathbf{y},\mathbf{z}) d\mathbf{z} d\mathbf{y} \tag{4}$$

for $k \ge 2$ where

$$h_2(\mathbf{y},\mathbf{z}) = \lambda^2 g(\mathbf{y}-\mathbf{z})[1-g(\mathbf{y}-\mathbf{x_0})]g(\mathbf{z}-\mathbf{x_0}), \tag{5}$$

$$h_k(\mathbf{y},\mathbf{z}) = \int_{\mathbb{R}^2} \lambda g(\mathbf{y}-\mathbf{z})[1-g(\mathbf{y}-\mathbf{w})]h_{k-1}(\mathbf{z},\mathbf{w}) d\mathbf{w}. \tag{6}$$

Proof: Imagine we partition the plane $\mathbb{R}^2$ into small and non-overlapping square areas of side length Δ. Assume that one of the square areas is centered at $\mathbf{x_0}$. Then we have the collection of square areas centered at $\mathbb{D}^2 = \{\mathbf{x_0} + (\mathbf{v} \cdot \Delta) : \mathbf{v} \in \mathbb{Z}^2\}$. Denote by $B_{\mathbf{x}}$ the square centered at $\mathbf{x}$. The area of $B_{\mathbf{x}}$ is $\delta\mathbf{x} = |B_{\mathbf{x}}| = \Delta^2$. Since nodes are Poissonly distributed in $\mathbb{R}^2$, the probability that there exists exactly one node within $B_{\mathbf{x}}$ is $p_1(B_{\mathbf{x}}) = \lambda\delta\mathbf{x} + o(\delta\mathbf{x})$ where $o(\delta\mathbf{x})$ denotes a quantity which, for small $\delta\mathbf{x}$, is of lower order than $\delta\mathbf{x}$, i.e. $\lim_{\delta\mathbf{x}\to 0} \frac{o(\delta\mathbf{x})}{\delta\mathbf{x}} = 0$. The probability that there is more than one node in $B_{\mathbf{x}}$ is $o(\delta\mathbf{x})$. Let $I_{\mathbf{x}}^k$ be the indicator random variable of the event that there exists exactly one node within the square area $B_{\mathbf{x}}$ *and* the node is a k-hop node. It follows from the above definition that

$$E[N_k] = \lim_{\Delta\to 0} \sum_{\mathbf{x}\in\mathbb{D}^2\setminus\{\mathbf{x_0}\}} E\left[I_{\mathbf{x}}^k\right] = \lim_{\Delta\to 0} \sum_{\mathbf{x}\in\mathbb{D}^2\setminus\{\mathbf{x_0}\}} \Pr\left\{I_{\mathbf{x}}^k = 1\right\}. \quad (7)$$

Similarly, let $J_{\mathbf{x}}$ be the indicator random variable of the event that there exists exactly one node within $B_{\mathbf{x}}$. Without loss of generality, we assume that when $J_{\mathbf{x}} = 1$, the node within $B_{\mathbf{x}}$ is located at $\mathbf{x}$. The difference between the actual location of the node within $B_{\mathbf{x}}$ and $\mathbf{x}$ becomes negligibly small as $\Delta \to 0$. Let $H_{\mathbf{x},\mathbf{y}}$ be the indicator random variable of the event that a node at $\mathbf{x}$ is directly connected to another node at $\mathbf{y}$. Recall that the connection has been assumed to be symmetric, i.e. $H_{\mathbf{x},\mathbf{y}} = H_{\mathbf{y},\mathbf{x}}$.

The probability that a 1-hop node exists in $B_{\mathbf{y}}$ is

$$\begin{aligned}\Pr\left\{I_{\mathbf{y}}^1 = 1\right\} =& \Pr\left\{J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_0}} = 1\right\} \\ =& g(\mathbf{y} - \mathbf{x_0}) \times [\lambda\delta\mathbf{y} + o(\delta\mathbf{y})]. \quad (8)\end{aligned}$$

Using Eq. (7) and (8) we obtain

$$E[N_1] = \lim_{\Delta\to 0} \sum_{\mathbf{y}\in\mathbb{D}^2\setminus\{\mathbf{x_0}\}} \Pr\left\{I_{\mathbf{y}}^1 = 1\right\} = \int_{\mathbb{R}^2} \lambda g(\mathbf{y} - \mathbf{x_0}) d\mathbf{y} \quad (9)$$

which is Eq. (3). A node is a 2-hop node if it is directly connected to *at least* one of the 1-hop nodes but not the node at $\mathbf{x_0}$. By applying the union bound and with some arithmetic steps we obtain

$$\begin{aligned}E[N_2] =& \lim_{\Delta\to 0} \sum_{\mathbf{y}\in\mathbb{D}^2\setminus\{\mathbf{x_0}\}} \Pr\left\{I_{\mathbf{y}}^2 = 1\right\} \\ \le& \lim_{\Delta\to 0} \sum_{\mathbf{y}\in\mathbb{D}^2\setminus\{\mathbf{x_0}\}} \sum_{\mathbf{x_1}\in\mathbb{D}^2\setminus\{\mathbf{y},\mathbf{x_0}\}} \Big[\Pr\left\{I_{\mathbf{x_1}}^1 = 1\right\} \\ &\times \Pr\left\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_1}} = 1, H_{\mathbf{y},\mathbf{x_0}} = 0 \middle| I_{\mathbf{x_1}}^1 = 1\right\}\Big] \\ &\text{[see Appendix A]} \quad (10) \\ =& \lim_{\Delta\to 0} \sum_{\mathbf{y}\in\mathbb{D}^2\setminus\{\mathbf{x_0}\}} \sum_{\mathbf{x_1}\in\mathbb{D}^2\setminus\{\mathbf{y},\mathbf{x_0}\}} \big[g(\mathbf{x_1} - \mathbf{x_0})\left[\lambda\delta\mathbf{x_1} + o(\delta\mathbf{x_1})\right] \\ &\times g(\mathbf{y} - \mathbf{x_1})\left[1 - g(\mathbf{y} - \mathbf{x_0})\right]\left[\lambda\delta\mathbf{y} + o(\delta\mathbf{y})\right]\big] \\ =& \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \lambda^2 g(\mathbf{y} - \mathbf{x_1})\left[1 - g(\mathbf{y} - \mathbf{x_0})\right] g(\mathbf{x_1} - \mathbf{x_0}) d\mathbf{x_1} d\mathbf{y} \quad (11)\end{aligned}$$

and hence Eq. (4) is proved for $k = 2$. It can be shown that $h_2(\mathbf{y}, \mathbf{z})$ in Eq. (5) has the meaning of being an upper bound on the pdf (probability density function) of the existence of a 2-hop node at $\mathbf{y}$ *and* a 1-hop node at $\mathbf{z}$ *and* they are directly connected. By recursion, $h_k(\mathbf{y}, \mathbf{z})$ in Eq. (6) has the meaning of being an upper bound on the pdf of the existence of a k-hop node at $\mathbf{y}$ *and* a $(k-1)$-hop node at $\mathbf{z}$ *and* they are directly connected. Note that a node is a k-hop node if it is directly connected to at least one $(k-1)$-hop node and not directly connected to any of the i-hop nodes where $i < k-1$. Therefore using the union bound (with some arithmetic steps) and only considering the $(k-1)$-hop nodes and $(k-2)$-hop nodes, we obtain

$$\begin{aligned}E[N_k] =& \lim_{\Delta\to 0} \sum_{\mathbf{y}\in\mathbb{D}^2\setminus\{\mathbf{x_0}\}} \Pr\left\{I_{\mathbf{y}}^k = 1\right\} \\ \le& \lim_{\Delta\to 0} \sum_{\mathbf{y}\in\mathbb{D}^2\setminus\{\mathbf{x_0}\}} \sum_{\substack{\mathbf{x_{k-1}},\mathbf{x_{k-2}}\in\mathbb{D}^2\setminus\{\mathbf{y},\mathbf{x_0}\} \\ \mathbf{x_{k-1}}\neq\mathbf{x_{k-2}}}} \Big[\Pr\Big\{J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_{k-1}}} = 1, \\ & H_{\mathbf{y},\mathbf{x_{k-2}}} = 0 \Big| I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}}}^{k-2} = 1, H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}}} = 1\Big\} \\ &\times \Pr\left\{I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}}}^{k-2} = 1, H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}}} = 1\right\}\Big] \\ &\text{[See Appendix B]} \quad (12) \\ \le& \lim_{\Delta\to 0} \sum_{\mathbf{y}\in\mathbb{D}^2\setminus\{\mathbf{x_0}\}} \sum_{\substack{\mathbf{x_{k-1}},\mathbf{x_{k-2}}\in\mathbb{D}^2\setminus\{\mathbf{y},\mathbf{x_0}\} \\ \mathbf{x_{k-1}}\neq\mathbf{x_{k-2}}}} \left[g(\mathbf{y} - \mathbf{x_{k-1}})\left[1 - g(\mathbf{y} - \mathbf{x_{k-2}})\right]\right. \\ &\times \left.\left[\lambda\delta\mathbf{y} + o(\delta\mathbf{y})\right] h_{k-1}(\mathbf{x_{k-1}}, \mathbf{x_{k-2}})\right] \delta\mathbf{x_{k-2}}\delta\mathbf{x_{k-1}} \\ =& \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} h_k(\mathbf{y}, \mathbf{x_{k-1}}) d\mathbf{x_{k-1}} d\mathbf{y} \quad (13)\end{aligned}$$

and hence Eq. (4) is proved for generic k. ■

Theorem 2. *For $G(\mathcal{H}_{\lambda,\mathbf{x_0}}; g)$ the critical density λ_c is lower bounded by*

$$\lambda_c \ge \sqrt{1/f_{\sup}} \quad (14)$$

where

$$\begin{aligned}f_{\sup} = \sup_{\mathbf{w},\mathbf{x}\in\mathbb{R}^2} \Big\{&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(\mathbf{y} - \mathbf{z})\left[1 - g(\mathbf{y} - \mathbf{w})\right] \\ &\times g(\mathbf{z} - \mathbf{w})\left[1 - g(\mathbf{z} - \mathbf{x})\right] d\mathbf{y} d\mathbf{z}\Big\} \quad (15)\end{aligned}$$

Proof: From Eq. (6), we have

$$\begin{aligned}h_k(\mathbf{y}, \mathbf{z}) =& \int_{\mathbb{R}^2} \lambda g(\mathbf{y} - \mathbf{z})[1 - g(\mathbf{y} - \mathbf{w})] h_{k-1}(\mathbf{z}, \mathbf{w}) d\mathbf{w} \\ =& \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \lambda g(\mathbf{y} - \mathbf{z})[1 - g(\mathbf{y} - \mathbf{w})]\lambda g(\mathbf{z} - \mathbf{w}) \\ &\times [1 - g(\mathbf{z} - \mathbf{x})] h_{k-2}(\mathbf{w}, \mathbf{x}) d\mathbf{x} d\mathbf{w}. \quad (16)\end{aligned}$$

Using Eq. (15) and (16), it can be shown that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} h_k(\mathbf{y}, \mathbf{z}) d\mathbf{y} d\mathbf{z} \le \lambda^2 f_{\sup} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} h_{k-2}(\mathbf{w}, \mathbf{x}) d\mathbf{x} d\mathbf{w}. \quad (17)$$

By defining

$$h_1(\mathbf{y}, \mathbf{z}) = \lambda g(\mathbf{y} - \mathbf{z}) \phi(\mathbf{z} - \mathbf{x_0}) \quad (18)$$

where $\phi(.)$ is the Dirac delta function and using Eq. (17) recursively, we have

$$E[N_k] \le \begin{cases} \left[\lambda^2 f_{\sup}\right]^{\frac{k-2}{2}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} h_2(\mathbf{w}, \mathbf{x}) d\mathbf{x} d\mathbf{w} & k \text{ is even,} \\ \left[\lambda^2 f_{\sup}\right]^{\frac{k-1}{2}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} h_1(\mathbf{w}, \mathbf{x}) d\mathbf{x} d\mathbf{w} & k \text{ is odd.} \end{cases} \quad (19)$$

Since $E[|\mathcal{W}|] = \sum_{k=1}^{\infty} E[N_k]$, it follows from Eq. (19) that $E[|\mathcal{W}|]$ is finite if $\lambda^2 f_{\text{sup}} < 1$. Given the fact that $E[|\mathcal{W}|]$ is finite implies $\theta(\lambda) = 0$ [3, p. 37], the result follows. ■

B. Upper Bound on λ_c

For the convenience in later discussion, define the connection function alternatively, via a function $\bar{g} : \mathbb{R}_+ \to [0,1]$ such that

$$\bar{g}(\|\mathbf{x}\|) = g(\mathbf{x}) \tag{20}$$

for any $\mathbf{x} \in \mathbb{R}^2$. The rotational property of g in (1) allows us to do this.

Let us partition the plane $\mathbb{R}^2$ into non-overlapping hexagons, where the distance between the centers of two neighboring hexagons is $a > 0$. We further partition each hexagon into 6 non-overlapping equilateral triangles, as shown in Fig. 1. Consider a hexagon and an equilateral triangle in the hexagon;

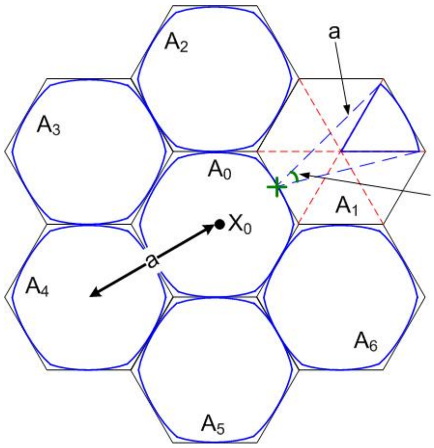

Fig. 1. The hexagons and the flower-shaped area in each hexagon.

there is exactly one hexagon side which is located directly opposite to the triangle. Centered at the middle point of that hexagon side, we draw a circle with radius a and obtain its intersectional area with the triangle. Repeat the action for the other 5 equilateral triangles in the hexagon. Merging the 6 intersectional areas we obtain a flower-shaped area within the hexagon. The flower-shaped area has an important property: any pair of nodes within two neighboring flower-shaped area have a maximum Euclidean distance of $2a$. Therefore, the probability that any two nodes inside two neighboring flower-shaped areas are directly connected is at least $\bar{g}(2a)$.

Starting with the flower-shaped area A_0 with a node X_0 located at the center, we examine the connections between X_0 and the nodes in the 6 neighboring flower-shaped areas, $A_1, A_2, \cdots, A_6$. We say A_i, $1 \leq i \leq 6$, is *occupied* if and only if there exists at least one node in A_i, *and* X_0 is directly connected to at least one of these nodes in A_i. The probability that A_i (with area $|A(a)|$) is occupied is then

$$p > \sum_{m=1}^{\infty} \left[\frac{[\lambda |A(a)|]^m \exp(-\lambda |A(a)|)}{m!} \times [1 - [1 - \bar{g}(2a)]^m] \right]$$

$$\begin{aligned} &= \left[1 - e^{-\lambda |A(a)|}\right] - e^{-\lambda |A(a)| \bar{g}(2a)} \left[1 - e^{-\lambda |A(a)|(1-\bar{g}(2a))}\right] \\ &= 1 - e^{-\lambda |A(a)| \bar{g}(2a)} \end{aligned} \tag{21}$$

where by elementary geometric calculation, $|A(a)|$ is given by $|A(a)| = 6\left[a^2\omega - \frac{a^2}{2}\sin(\omega)\right]$ with $\omega = \frac{\pi}{6} - \arcsin(\frac{1}{4})$. Note that the events that neighboring flower-shaped areas are occupied are independent. Next, for each occupied A_i we focus on a node $X_i \in A_i$ which is directly connected to X_0 and examine the direct connections from X_i to other nodes in the neighboring flower-shaped areas that have not been considered before. The process continues in the similar way.

The process above shows that $\Pr\{|\mathcal{W}| = \infty\} > 0$ *if* the probability that there are infinite number of flower-shaped areas which are occupied is positive, namely the flower-shaped areas percolate with positive probability. Note that the percolation of the flower-shaped areas implies site percolation on the accompanying equilateral triangular lattice, and vice versa. Hence, the flower-shaped areas percolates if $p > 0.5$ [4, p. 132]. That is, $\Pr\{|\mathcal{W}| = \infty\} > 0$ if

$$1 - e^{-\lambda |A(a)| \bar{g}(2a)} > 0.5 \Leftrightarrow \lambda > \frac{\log_e(2)}{|A(a)|\bar{g}(2a)}. \tag{22}$$

Indeed, $\Pr\{|\mathcal{W}| = \infty\} > 0$ if Eq. (22) holds *for any* value of a. The above analysis can be summarized into the following theorem.

Theorem 3. *For $G(\mathcal{H}_{\lambda,\mathbf{x_0}}; g)$ the critical density λ_c is upper bounded by*

$$\lambda_c \leq \inf_{a \in \mathbb{R}_+} \left\{ \frac{\log_e(2)}{|A(a)|\bar{g}(2a)} \right\} \tag{23}$$

where $|A(a)| = 6\left[a^2\omega - \frac{a^2}{2}\sin(\omega)\right]$ with $\omega = \frac{\pi}{6} - \arcsin(\frac{1}{4})$, $\bar{g}(\|\mathbf{x}\|) = g(\mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^2$.

IV. Discussion

In this section, we obtain the bounds on λ_c under the unit disk model as a special case of the results obtained in the previous section for the random connection model. Apply Eq. (2) into Theorem 2, we have $\lambda_c \geq \sqrt{1/f_{\text{sup}}}$ with

$$\begin{aligned} f_{\text{sup}} &= \int_{\|\mathbf{z}\| \leq r} \int_{\|\mathbf{y}-\mathbf{z}\| \leq r, \|\mathbf{y}\| > r} d\mathbf{y} d\mathbf{z} \\ &= \int_0^r \left[\int_{r-u}^r 2v \arccos\left(\frac{r^2 - u^2 - v^2}{2uv}\right) dv \right] 2\pi u du \end{aligned} \tag{24}$$

which follows from an elementary trigonometric calculation. Next, apply Eq. (2) into Theorem 3, it can be shown that the infimum is achieved at $a = \frac{1}{2}r$. That is,

$$\lambda_c \leq \frac{\log_e(2)}{|A(r/2)|} = \frac{\log_e(2)}{6\left[\frac{r^2}{4}\omega - \frac{r^2}{8}\sin(\omega)\right]} \tag{25}$$

with $\omega = \frac{\pi}{6} - \arcsin(\frac{1}{4})$.

As a specific example with $r = 2$, we can calculate numerically $f_{\text{sup}} \approx 65.345$ and $A(1) \approx 0.8227$. Hence, $0.124 \leq \lambda_c \leq 0.843$. This result is comparable with the results in the literature, which include the analytical bounds

$0.174 < \lambda_c < 0.843$ given by Meester and Roy [1], $0.2554 < \lambda_c < 0.843$ given by Gu and Hong [6], and $\lambda_c > 0.1925$ given by Kong and Yeh [7]. The upper bound we obtained is comparable with the upper bound obtained in [1], [6]. Although the lower bounds obtained in those papers are tighter than our lower bound, their results are valid for the unit disk model only. An extension of their work (using the methods of those papers) to the random connection model could appear to be non-trivial.

V. Conclusions and Future Work

In this paper we investigated the analytical bounds on the critical density in 2D wireless multi-hop networks where nodes are Poissonly distributed in the plane $\mathbb{R}^2$. The establishment of direct connection between two nodes follows a random connection model satisfying some intuitively reasonable conditions, i.e. rotational and translation invariance, non-increasing monotonicity and integral boundedness. We obtained a lower bound on the critical density using a Galton-Watson branching process and an upper bound by coupling the problem to site percolation in triangular lattice. From the results, we then obtained the bounds under the unit disk communication model, which is a special case of the random connection model, and compared them with the existing results in the literature. The outcome showed that our method generates bounds close to existing results in the literature. In future work, we plan to extend the current work along two directions: a) consider other non-Poisson node distributions, b) tighten the bounds obtained in the paper.

Appendix A
Derivations of Eq. (10)

In this section, we follow the definitions and notations used in Lemma 1. Note that if a node Y is a 2-hop node, then it is not directly connected to the node at $\mathbf{x_0}$ but directly connected to at least one 1-hop nodes. Recall that N_1 is a random variable denoting the number of 1-hop nodes, hence for any positive integer n_1 we have

$$\begin{aligned}
&\Pr\left\{ I_{\mathbf{y}}^2 = 1, \bigcap_{i=1}^{n_1} I_{\mathbf{x_1^i}}^1 = 1 \,\middle|\, N_1 = n_1 \right\} \\
&= \Pr\left\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_0}} = 0, \bigcup_{i=1}^{n_1} H_{\mathbf{y},\mathbf{x_1^i}} = 1, \bigcap_{i=1}^{n_1} I_{\mathbf{x_1^i}}^1 = 1 \,\middle|\, N_1 = n_1 \right\} \\
&\le \sum_{j=1}^{n_1} \Pr\left\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_0}} = 0, H_{\mathbf{y},\mathbf{x_1^j}} = 1, \bigcap_{i=1}^{n_1} I_{\mathbf{x_1^i}}^1 = 1 \,\middle|\, N_1 = n_1 \right\} \\
&\qquad \text{[union bound]}. \qquad (26)
\end{aligned}$$

The conditional probability $\Pr\left\{ I_{\mathbf{y}}^2 = 1 \middle| N_1 = n_1 \right\}$ is then obtained as follows.

$$\begin{aligned}
&\Pr\left\{ I_{\mathbf{y}}^2 = 1 \middle| N_1 = n_1 \right\} \\
&= \frac{1}{n_1!} \sum_{\substack{\mathbf{x_1^1}, \cdots, \mathbf{x_1^{n_1}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\} \\ \mathbf{x_1^m} \neq \mathbf{x_1^n} \text{ for } m \neq n}} \Pr\left\{ I_{\mathbf{y}}^2 = 1, \bigcap_{i=1}^{n_1} I_{\mathbf{x_1^i}}^1 = 1 \,\middle|\, N_1 = n_1 \right\} \\
&\le \frac{1}{n_1!} \sum_{\substack{\mathbf{x_1^1}, \cdots, \mathbf{x_1^{n_1}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\} \\ \mathbf{x_1^m} \neq \mathbf{x_1^n} \text{ for } m \neq n}} \sum_{j=1}^{n_1} \Pr\Big\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_0}} = 0, \\
&\qquad H_{\mathbf{y},\mathbf{x_1^j}} = 1, \bigcap_{i=1}^{n_1} I_{\mathbf{x_1^i}}^1 = 1 \,\Big|\, N_1 = n_1 \Big\} \qquad \text{[from Eq. (26)]} \\
&= \frac{1}{n_1!} \sum_{j=1}^{n_1} \sum_{\mathbf{x_1^j} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\}} \Bigg[\sum_{\substack{\mathbf{x_1^1}, \cdots, \mathbf{x_1^{j-1}}, \mathbf{x_1^{j+1}}, \cdots, \mathbf{x_1^{n_1}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0},\mathbf{x_1^j}\} \\ \mathbf{x_1^m} \neq \mathbf{x_1^n} \text{ for } m \neq n}} \Pr\Big\{ J_{\mathbf{y}} = 1, \\
&\qquad H_{\mathbf{y},\mathbf{x_0}} = 0, H_{\mathbf{y},\mathbf{x_1^j}} = 1, \bigcap_{i=1}^{n_1} I_{\mathbf{x_1^i}}^1 = 1 \,\Big|\, N_1 = n_1 \Big\} \Bigg] \\
&\qquad \text{[move the summation on } j \text{ and } \mathbf{x_1^j} \text{ to the outermost]} \\
&= \frac{1}{n_1!} \sum_{j=1}^{n_1} \sum_{\mathbf{x_1^j} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\}} \Big[(n_1 - 1)! \Pr\Big\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_0}} = 0, \\
&\qquad H_{\mathbf{y},\mathbf{x_1^j}} = 1, I_{\mathbf{x_1^j}}^1 = 1 \,\Big|\, N_1 = n_1 \Big\} \Big] \\
&\qquad \text{[resolve the summations inside the square brackets]} \\
&= \sum_{\mathbf{x_1} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\}} \Big[\Pr\Big\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_0}} = 0, H_{\mathbf{y},\mathbf{x_1}} = 1 \,\Big| \\
&\qquad I_{\mathbf{x_1}}^1 = 1, N_1 = n_1 \Big\} \Pr\Big\{ I_{\mathbf{x_1}}^1 = 1 \,\Big|\, N_1 = n_1 \Big\} \Big]. \qquad (27)
\end{aligned}$$

Since

$$\Pr\left\{ I_{\mathbf{x_1}}^1 = 1 \middle| N_1 = n_1 \right\} = \frac{\Pr\left\{ I_{\mathbf{x_1}}^1 = 1, N_1 = n_1 \right\}}{\Pr\left\{ N_1 = n_1 \right\}} \qquad (28)$$

and

$$\sum_{n_1=1}^{\infty} \Pr\left\{ I_{\mathbf{x_1}}^1 = 1, N_1 = n_1 \right\} = \Pr\left\{ I_{\mathbf{x_1}}^1 = 1 \right\}, \qquad (29)$$

using Eq. (27), (28) and (29) we obtain

$$\begin{aligned}
&\Pr\left\{ I_{\mathbf{y}}^2 = 1 \right\} \\
&= \sum_{n_1=1}^{\infty} \left[\Pr\left\{ I_{\mathbf{y}}^2 = 1 \middle| N_1 = n_1 \right\} \Pr\left\{ N_1 = n_1 \right\} \right] \\
&\le \sum_{\mathbf{x_1} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\}} \Big[\Pr\Big\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_0}} = 0, H_{\mathbf{y},\mathbf{x_1}} = 1 \,\Big|\, I_{\mathbf{x_1}}^1 = 1 \Big\} \\
&\quad \times \Pr\left\{ I_{\mathbf{x_1}}^1 = 1 \right\} \Big]. \qquad (30)
\end{aligned}$$

Hence we obtain Eq. (10).

Appendix B
Derivations of Eq. (12)

In this section, we follow the definitions and notations used in Lemma 1. The upper bound on the probability that a k-hop node exists within $B_{\mathbf{y}}$ conditioned on that $N_{k-2} = n_{k-2}$ is obtained as follows.

$$\begin{aligned}
&\Pr\left\{ I_{\mathbf{y}}^k = 1 \middle| N_{k-2} = n_{k-2} \right\} \\
&= \frac{1}{n_{k-2}!} \sum_{\substack{\mathbf{x_{k-2}^1}, \cdots, \mathbf{x_{k-2}^{n_{k-2}}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\} \\ \mathbf{x_{k-2}^m} \neq \mathbf{x_{k-2}^n} \text{ for } m \neq n}} \Pr\left\{ I_{\mathbf{y}}^k = 1, \bigcap_{i=1}^{n_{k-2}} I_{\mathbf{x_{k-2}^i}}^{k-2} = 1 \,\right|
\end{aligned}$$

$$N_{k-2} = n_{k-2}\} . \quad (31)$$

By generalizing the derivation of Eq. (30), it can be shown that,

$$\Pr\left\{ I_{\mathbf{y}}^{k} = 1, \bigcap_{i=1}^{n_{k-2}} I_{\mathbf{x_{k-2}^{i}}}^{k-2} = 1 \middle| N_{k-2} = n_{k-2} \right\}$$
$$\leq \sum_{\mathbf{x_{k-1}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\}} \Pr\left\{ J_{\mathbf{y}} = 1, \bigcap_{i=1}^{n_{k-2}} H_{\mathbf{y},\mathbf{x_{k-2}^{i}}} = 0, H_{\mathbf{y},\mathbf{x_{k-1}}} = 1, \right.$$
$$\bigcup_{i=1}^{n_{k-2}} H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}^{i}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, \bigcap_{i=1}^{n_{k-2}} I_{\mathbf{x_{k-2}^{i}}}^{k-2} = 1 \Bigg|$$
$$N_{k-2} = n_{k-2}\}$$
$$\leq \sum_{j=1}^{n_{k-2}} \sum_{\mathbf{x_{k-1}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\}} \Pr\Big\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_{k-2}^{j}}} = 0, H_{\mathbf{y},\mathbf{x_{k-1}}} = 1,$$
$$H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}^{j}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, \bigcap_{i=1}^{n_{k-2}} I_{\mathbf{x_{k-2}^{i}}}^{k-2} = 1 \Bigg|$$
$$N_{k-2} = n_{k-2}\} \qquad \text{[union bound]}$$
$$= \sum_{j=1}^{n_{k-2}} \sum_{\mathbf{x_{k-1}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\}} \Big[\Pr\Big\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_{k-2}^{j}}} = 0, H_{\mathbf{y},\mathbf{x_{k-1}}} = 1 \Big|$$
$$\left. H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}^{j}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}^{j}}}^{k-2} = 1, N_{k-2} = n_{k-2} \right\}$$
$$\times \Pr\left\{ H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}^{j}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, \bigcap_{i=1}^{n_{k-2}} I_{\mathbf{x_{k-2}^{i}}}^{k-2} = 1 \right|$$
$$N_{k-2} = n_{k-2}\}\Big] . \quad (32)$$

Note that for any j,

$$\sum_{\substack{\mathbf{x_{k-2}^{1}},\cdots,\mathbf{x_{k-2}^{j-1}},\mathbf{x_{k-2}^{j+1}},\cdots,\mathbf{x_{k-2}^{n_{k-2}}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0},\mathbf{x_{k-1}},\mathbf{x_{k-2}^{j}}\} \\ \mathbf{x_{k-2}^{m}} \neq \mathbf{x_{k-2}^{n}} \text{ for } m \neq n}} \Pr\Big\{ H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}^{j}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1,$$
$$\left. \bigcap_{i=1}^{n_{k-2}} I_{\mathbf{x_{k-2}^{i}}}^{k-2} = 1 \middle| N_{k-2} = n_{k-2} \right\}$$
$$= (n_{k-2} - 1)! \Pr\Big\{ H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}^{j}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1,$$
$$\left. I_{\mathbf{x_{k-2}^{j}}}^{k-2} = 1 \middle| N_{k-2} = n_{k-2} \right\} . \quad (33)$$

Substitute Eq. (32) and (33) into Eq. (31),

$$\Pr\left\{ I_{\mathbf{y}}^{k} = 1 \middle| N_{k-2} = n_{k-2} \right\}$$
$$\leq \frac{1}{n_{k-2}!} \sum_{j=1}^{n_{k-2}} \sum_{\substack{\mathbf{x_{k-1}},\mathbf{x_{k-2}^{j}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\} \\ \mathbf{x_{k-1}} \neq \mathbf{x_{k-2}^{j}}}} \Big[\Pr\Big\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_{k-2}^{j}}} = 0,$$
$$H_{\mathbf{y},\mathbf{x_{k-1}}} = 1 \Big| H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}^{j}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}^{j}}}^{k-2} = 1,$$
$$N_{k-2} = n_{k-2}\} \times (n_{k-2} - 1)! \Pr\Big\{ H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}^{j}}} = 1,$$
$$\left. I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}^{j}}}^{k-2} = 1 \middle| N_{k-2} = n_{k-2} \right\} \Big]$$

$$= \sum_{\substack{\mathbf{x_{k-1}},\mathbf{x_{k-2}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\} \\ \mathbf{x_{k-1}} \neq \mathbf{x_{k-2}}}} \Big[\Pr\Big\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_{k-2}}} = 0, H_{\mathbf{y},\mathbf{x_{k-1}}} = 1 \Big|$$
$$\left. H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}}}^{k-2} = 1, N_{k-2} = n_{k-2} \right\}$$
$$\times \Pr\Big\{ H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}}}^{k-2} = 1 \Big|$$
$$N_{k-2} = n_{k-2}\}\Big] . \quad (34)$$

Since

$$\Pr\left\{ H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}}}^{k-2} = 1 \middle| N_{k-2} = n_{k-2} \right\}$$
$$= \frac{\Pr\left\{ H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}}}^{k-2} = 1, N_{k-2} = n_{k-2} \right\}}{\Pr\{N_{k-2} = n_{k-2}\}} \quad (35)$$

and

$$\sum_{n_{k-2}=1}^{\infty} \Pr\Big\{ H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}}}^{k-2} = 1,$$
$$N_{k-2} = n_{k-2}\}$$
$$= \Pr\left\{ H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}}}^{k-2} = 1 \right\} , \quad (36)$$

using Eq. (34), (35) and (36) we obtain

$$\Pr\left\{ I_{\mathbf{y}}^{k} = 1 \right\}$$
$$= \sum_{n_{k-2}=1}^{\infty} \left[\Pr\left\{ I_{\mathbf{y}}^{k} = 1 \middle| N_{k-2} = n_{k-2} \right\} \Pr\{N_{k-2} = n_{k-2}\} \right]$$
$$\leq \sum_{\substack{\mathbf{x_{k-1}},\mathbf{x_{k-2}} \in \mathbb{D}^2 \backslash \{\mathbf{y},\mathbf{x_0}\} \\ \mathbf{x_{k-1}} \neq \mathbf{x_{k-2}}}} \Big[\Pr\Big\{ J_{\mathbf{y}} = 1, H_{\mathbf{y},\mathbf{x_{k-2}}} = 0, H_{\mathbf{y},\mathbf{x_{k-1}}} = 1 \Big|$$
$$\left. H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}}}^{k-2} = 1 \right\}$$
$$\times \Pr\left\{ H_{\mathbf{x_{k-1}},\mathbf{x_{k-2}}} = 1, I_{\mathbf{x_{k-1}}}^{k-1} = 1, I_{\mathbf{x_{k-2}}}^{k-2} = 1 \right\} \Big] . \quad (37)$$

Hence we obtain Eq. (12).

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