A Necessary Condition for Connected Wireless CSMA Multi-hop Networks

Tao Yang, Guoqiang Mao
School of Electrical and Information Engineering
The University of Sydney
Email: tao.yang, guoqiang.mao@sydney.edu.au

Wei Zhang
School of Electrical Engineering and Telecommunications
The University of New South Wales
Email: wzhang@ee.unsw.edu.au

***Abstract*—Connectivity is one of the most fundamental properties of wireless multi-hop networks. In a wireless network with many concurrent transmissions, signals transmitted at the same time may mutually interfere with each other. In this paper we consider the impact of interference on the connectivity of CSMA networks using the SINR model.**

On the basis of our earlier work in which we give a sufficient condition, i.e. an upper bound, on the critical transmission power required for a CSMA network with a total of n nodes i.i.d. on a $\sqrt{n}\times\sqrt{n}$ square following a uniform distribution to be *a.a.s.* connected as $n\to\infty$ under the SINR model, in this paper we continue to study the necessary condition for the above CSMA network to be *a.a.s.* connected. A lower bound is obtained on the critical transmission power required for the above CSMA network to be *a.a.s.* connected under *any* scheduling scheme satisfying the carrier-sensing constraint. The lower bound differs from the upper bound by a constant factor only.

Compared with previous literature assuming a unit disk model, it is shown that the critical transmission power for a CSMA network under the SINR model to be *a.a.s.* connected is within a constant factor of that required for a network under the unit disk model, which does not consider the impact of interference, to be *a.a.s.* connected. That is, transmission power only needs to be increased by a constant factor to combat interference and maintain connectivity. This result is also in sharp contrast with previous results considering the connectivity of ALOHA networks under the SINR model.

I. INTRODUCTION

Connectivity is a prerequisite in wireless multi-hop networks for providing most network functions (e.g. routing, localization and topology control) [1], [2], [3]. A wireless multi-hop network is said to be *connected* if and only if (iff) there is at least one (multi-hop) path between any pair of nodes in the network.

Due to the nature of wireless communications, signals transmitted at the same time may mutually interfere with each other. The SINR (signal-to-noise-plus-interference-ratio) model has been widely used to capture the impact of interference on network connectivity [4], [5], [2]. Under the SINR model, the existence of a directional link between a pair of nodes is determined by the strength of the received signal from desired transmitter, the interference caused by other concurrent transmissions and the background noise. Assume that all nodes transmit with the same power P and let $\boldsymbol{x}_k$, $k\in\Gamma$ be the location of node k, where Γ represents the set of indices of all nodes in the network. A node j can successfully receive the transmitted signal from a node i (i.e. node j *is directly connected to* node i) if the SINR at $\boldsymbol{x}_j$, denoted by $\mathrm{SINR}\left(\boldsymbol{x}_i\to\boldsymbol{x}_j\right)$, is above a prescribed threshold β, i.e.

$$\mathrm{SINR}\left(\boldsymbol{x}_i\to\boldsymbol{x}_j\right)=\frac{P\ell\left(\boldsymbol{x}_i,\,\boldsymbol{x}_j\right)}{N_0+\gamma\sum\limits_{k\in\mathcal{T}_i}P\ell\left(\boldsymbol{x}_k,\,\boldsymbol{x}_j\right)}\geq\beta \quad (1)$$

where $\mathcal{T}_i\subseteq\Gamma$ denotes the subset of nodes that are transmitting at the same time as node i and N_0 is the background noise power. The function $\ell\left(\boldsymbol{x}_i,\,\boldsymbol{x}_j\right)$ is the power attenuation from $\boldsymbol{x}_i$ to $\boldsymbol{x}_j$. The coefficient $0\leq\gamma\leq 1$ is the inverse of the processing gain of the system and it weighs the impact of interference. In a broadband system using CDMA, γ depends on the orthogonality between codes used during concurrent transmissions and is smaller than 1; in a narrow-band system, γ equals to 1 [2], [5].

Similarly, node i can receive from node j (i.e. node i is directly connected to node j) iff

$$\mathrm{SINR}\left(\boldsymbol{x}_j\to\boldsymbol{x}_i\right)=\frac{P\ell\left(\boldsymbol{x}_j,\,\boldsymbol{x}_i\right)}{N_0+\gamma\sum\limits_{k\in\mathcal{T}_j}P\ell\left(\boldsymbol{x}_k,\,\boldsymbol{x}_i\right)}\geq\beta. \quad (2)$$

Therefore node i and node j are directly connected, i.e. a bidirectional link exists between node i and node j, iff both (1) and (2) are satisfied.

In [5] Dousse *et al.* use the above SINR model to analyze the impact of interference on connectivity from the percolation perspective. They consider a network where all nodes are distributed in $\Re^2$ following a homogeneous Poisson point process with a constant density λ and an attenuation function ℓ with bounded support. By letting $\mathcal{T}_j=\Gamma/\left\{i,j\right\}$, i.e. all other nodes in the network transmit simultaneously with node i irrespective of their relative locations to $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, it is shown that there exist γ' such that if $\gamma>\gamma'$ there is no infinite connected component in the network, i.e. the network does not *percolate*. However when $\gamma<\gamma'$, there exists $0<\lambda'<\infty$ such that percolation can still occur when $\lambda>\lambda'$. An improved result by the same authors in [6] shows that under the more general conditions that $\lambda>\lambda_c$ and the attenuation function has unbounded support, percolation occurs when $\gamma<\gamma'$. Here λ_c is the critical node density above which the network with $\gamma=0$ (i.e. *the unit disk model*) percolates [7, p48]. The above results suggest that percolation under the SINR model can happen iff γ is sufficiently small. In their network setting, it is

This research is funded by ARC Discovery project: DP110100538.

assumed that each node transmits randomly and independently, irrespective of any nearby transmitter. This corresponds to the ALOHA-type multiple access scheme [2]. This ALOHA multiple access strategy however has become obsolete [8], [9].

The ALOHA-type multiple access strategy has been superceded by more advanced multiple access strategies, e.g. CSMA and CSMA/CD (Carrier Sense Multiple Access with Collision Detection) [10]. The general idea of CSMA schemes is that nearby nodes will not be scheduled to be simultaneously active, *i.e.* the Euclidean distance between any two concurrent transmitters will not be less than a threshold. CSMA schemes have been shown to improve the performance of ALOHA schemes by alleviating interference, particularly under heavy traffic.

In [11] we provide a sufficient condition required for a CSMA network with n nodes *i.i.d.* on a $\sqrt{n} \times \sqrt{n}$ square following a uniform distribution to be *a.a.s.* (asymptotically almost surely) connected as $n \to \infty$. [An event ξ_n, depending on n is said to occur *a.a.s.* if its probability tends to one as $n \to \infty$.]

In this paper, we continue to study the necessary condition required for the above CSMA network [11] to be *a.a.s.* connected as $n \to \infty$ under SINR model, viz. the necessary condition required for the above network with n nodes *i.i.d.* on a $\sqrt{n} \times \sqrt{n}$ square following a uniform distribution to be *a.a.s.* connected under *any* scheduling scheme satisfying the carrier-sensing constraint as $n \to \infty$.

The rest of this paper is organized as follows. Section II reviews related work. Section III defines system models. Section IV describes a particular scheduling algorithm studied in the paper and its relation to the necessary condition. In Section V we give a necessary condition for CSMA networks under the scheduling algorithm described in Section IV to be *a.a.s.* connected under SINR model. Finally Section VI concludes the paper and sets future work.

II. Related Work

Extensive research has been done on connectivity problems using the well-know random geometric graph and the *unit disk model*, which is usually obtained by randomly and uniformly distributing n nodes in a given area and connecting any two nodes iff their Euclidean distance is smaller than or equal to a certain threshold $r(n)$. This model corresponds to a special case of the SINR model in (1), i.e. when $\gamma = 0$ (perfect orthogonality, no mutual interference). It has been shown that under the unit disk model and in a disk of unit area, the above network with a transmission range of $r(n) = \sqrt{\frac{\log n + c(n)}{\pi n}}$ is *a.a.s.* connected as $n \to \infty$ iff $c(n) \to \infty$ [1], [12].

Other work in the area includes [13], [3], which investigate the necessary condition for the above network to be *a.a.s.* connected under the more realistic *log-normal connection model*. Under the log-normal connection model, two nodes are directly connected if the received power at one node from the other node, whose attenuation follows the log-normal model [10], is greater than a given threshold. These results [13], [3] however rely on the assumption that the event that a node is isolated and the event that another node is isolated are independent.

Despite the significant impact of interference caused by multiple concurrent transmissions on network connectivity, very limited work has been done on analyzing network connectivity under the SINR model. In [14], Avin *et al.* study connectivity from the perspective of channel assignment. Specifically, considering a two-dimensional grid network with a set of active links representing communication requests and assigning a color (time slots/frequencies) to each active link, they investigated how many colors are necessary in order for all active links to be simultaneously transmitting while satisfying the SINR requirement. In [15], Lebhar *et al.*, by enforcing each node to transmit in a particular time slot, show how to emulate a unit disk graph in which each link operates under the SINR constraint. The two papers [5], [6] discussed in Section I consider a network with nodes Poissonly distributed in $\Re^2$ under the SINR model and the ALOHA multiple access scheme, and give conditions under which the network percolates. In [11], we give a sufficient condition on the transmission power required for a CSMA network with n nodes *i.i.d.* on a $\sqrt{n} \times \sqrt{n}$ square following a uniform distribution to be *a.a.s.* connected as $n \to \infty$.

III. System Model

In this paper, we consider a network with n nodes i.i.d. in a square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ following a uniform distribution, i.e. the so-called *extended network model* [2]. It is assumed that all nodes use the same transmission power P and there is always a packet at a node waiting to be transmitted. This later assumption allows us to focus on the network property without being disturbed by other factors, e.g. traffic distribution.

Further, we assume that the attenuation function ℓ in (1) only depends on Euclidean distance, namely $\ell : \Re^+ \to \Re^+$, and is a power-law function [5], [6], [2]:

$$\ell(s) = s^{-\alpha} \tag{3}$$

where s represents the Euclidean distance between a pair of nodes and α is the path-loss exponent, which typically varies from 2 to 6 [10, p139]. In this paper we assume that $\alpha > 2$. The above assumptions on the attenuation function are widely used [2], [5], [16] and are also supported by measurement studies [10]. The results in this paper can be readily extended to the situation where $\ell(s) = \min\{1, s^{-\alpha}\}$ at the expense of some lengthy but mostly straightforward discussions on the special case when $s^{-\alpha} > 1$. As commonly done in the connectivity analysis [1], [5], [6], [7], [12], the impact of small-scale fading is ignored and only bidirectional communication links are considered. Further, since in dense sensor networks and cellular networks the background noise is typically negligibly small [2], [16], in the following analysis, we ignore the background noise N_0 in (1) and (2) to focus on the main ideas. The result can be easily extended to include the impact of background noise. In this paper, we consider that all transmitters are using the same channel, i.e. $\gamma = 1$, which corresponds to a narrow-band system [2], [5], [16].

In CSMA networks, two nodes are allowed to transmit at the same time iff they can not detect each other's transmission, i.e. both $P\ell(\boldsymbol{x}_i, \boldsymbol{x}_j)$ and $P\ell(\boldsymbol{x}_j, \boldsymbol{x}_i)$ in (1) and (2) are under a certain detection threshold P_{th}. It then follows from (3) that the carrier-sensing range R_c, which determines the *minimum* Euclidean distance between two concurrent transmitters, is given by

$$R_c = (P/P_{\text{th}})^{1/\alpha} \tag{4}$$

IV. Scheduling Algorithm

The focus of this paper is on obtaining a necessary condition, i.e. the minimum transmission power, required for a network with n nodes *i.i.d.* on a $\sqrt{n} \times \sqrt{n}$ square following a uniform distribution to be *a.a.s.* connected under *any* scheduling scheme satisfying carrier-sensing constraint as $n \to \infty$. Denote that minimum transmission power by P_Ω, where Ω represents the set of all scheduling algorithms satisfying the carrier-sensing constraint. Because a wireless multihop network is often operated under a distributed scheduling algorithm, any set of transmitters can be possibly active as long as they do not sense each other's transmission. Therefore it is important to set the transmission power of nodes to be above P_Ω to ensure connectivity.

Denote by $\omega \in \Omega$ a particular scheduling algorithm satisfying the carrier-sensing constraint and by P_ω the minimum transmission power required for the above network to be connected under the scheduling algorithm ω.

As an easy consequence of (4), as the transmission power increases, the carrier sensing range R_c will also increase. Therefore, other things being equal, the number of simultaneously active transmitters will decrease. It then follows from (1) and (2) (remember that $N_0 = 0$) that as the transmission power increases, if a pair of nodes are directly connected originally, they will remain connected. Therefore connectivity is a monotonic non-decreasing property of transmsion power. The minimum transmission power required for the network to be connected under *any* scheduling algorithm in Ω is necessarily larger than the minimum transmission power required for the network to be connected under a particular scheduling algorithm $\omega \in \Omega$, i.e.

$$P_\Omega = \max_{\omega \in \Omega} P_\omega \quad \text{and} \quad P_\Omega \geq P_\omega \tag{5}$$

On the basis of (5), the task now becomes finding a particular scheduling algorithm ω that gives as large P_ω as possible, i.e. finding a tight lower bound of P_Ω. In the following paragraphs, we construct such a scheduling algorithm ω heuristically.

Such an algorithm first needs to satisfy the constraint on the minimum Euclidean distance between concurrent transmitters imposed by the carrier-sensing requirement. In the meantime, the algorithm needs to schedule as many concurrent transmissions as possible to maximize interference, hence P_ω.

We first start with a technical lemma which is required for the construction of the scheduling algorithm.

Lemma 1. *Partition the square* $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ *into non-overlapping hexagons of equal side length* s_n *such that the origin* o *coincides with the centre of a hexagon and two diagonal vertices of this hexagon, whose Euclidean distance is* $2s_n$*, are located on* y *axis, as shown in Figure 1. Further, we call a hexagon an* interior hexagon *if it is entirely contained in the square* $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ *(an* exterior hexagon *is defined analogously) . When* $s_n = \sqrt{\frac{2\log n}{5}}$*, each interior hexagon is occupied by at least one node* a.a.s.

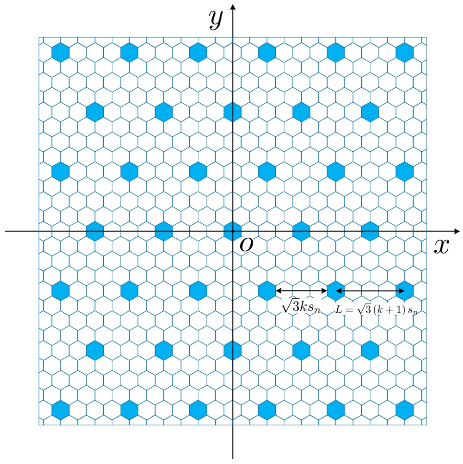

Figure 1. An illustration of the hexagonal partition of the network area. The shaded hexagons represent simultaneously active hexagons, where $k = 3$.

Proof: Because nodes are i.i.d. following a uniform distribution, for an arbitrary interior hexagon, the probability that it is empty is given by $\left(1 - \frac{3\sqrt{3}s_n^2}{2n}\right)^n$. Let ξ_i be the event that an interior hexagon i is empty, where $i \in \Xi$ and Ξ denotes the set of indices of all interior hexagons. Since each interior hexagon occupies an area of $\frac{3\sqrt{3}s_n^2}{2}$, there are at most $m = \frac{2n}{3\sqrt{3}s_n^2}$ interior hexagons. Therefore $|\Xi| \leq m$.

Denote by A_n the event that there is at least one empty interior hexagon in $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$. It follows that

$$\Pr(A_n) = \Pr\left(\bigcup_{i \in \Xi} \xi_i\right)$$

Using union bound, we have

$$\Pr\left(\bigcup_{i \in \Xi} \xi_i\right) \leq \sum_{i=1}^{m} \Pr(\xi_i) = \frac{2n\left(1 - \frac{3\sqrt{3}s_n^2}{2n}\right)^n}{3\sqrt{3}s_n^2}$$

Using the fact that $1 - x \leq \exp(-x)$ and $s_n = \sqrt{\frac{2\log n}{5}}$, we have

$$\begin{aligned}\lim_{n\to\infty} \Pr(A_n) &\leq \lim_{n\to\infty} \frac{2ne^{-\frac{3\sqrt{3}s_n^2}{2}}}{3\sqrt{3}s_n^2} \\ &= \lim_{n\to\infty} \frac{5n}{3\sqrt{3}n^{\frac{3\sqrt{3}}{5}}\log n} = 0\end{aligned}$$

Therefore, it can be concluded that *a.a.s.* every interior hexagon is occupied by at least one node as $n \to \infty$. ■

Hereinafter, we declare a hexagon to be *active* if there is a node transmitting in it. Due to the minimum distance constraint among CSMA transmitters, any two active hexagons should be separated by a minimum Euclidean distance (depending on the carrier-sensing range given in (4)). Obviously no two simultaneously active transmitters can be located inside a single hexagon due to the choice of s_n and the carrier sensing requirement.

Let k be the minimum number of inactive hexagons between two closest simultaneously active hexagons (see Figure 1). It can be easily shown that any two nodes inside the two hexagons is separated by an Euclidean distance of at least $\sqrt{3}ks_n$. Since the Euclidean distance between any two concurrent CSMA transmitters should be greater than or equal to R_c, we choose the value of k such that

$$\sqrt{3}ks_n \geq R_c \geq \sqrt{3}\,(k-1)\,s_n \tag{6}$$

We further define a *maximal independent set* to be the set of hexagons that a) includes as many hexagons as possible; and b) closest hexagons in the set are separated by exactly k adjacent hexagons. Figure 1 illustrates such a maximal independent set with $k = 3$. Note that the number of maximal independent sets is finite and depends on k only.

Now we are ready to define the key properties of the scheduling algorithm:

The *scheduling algorithm* ω is such that only nodes in different hexagons belonging to the same maximal independent set can be scheduled to be active at the same time. No nodes in the same hexagon can be scheduled to be simultaneously active. (Note that if an exterior hexagon has node(s) in it, it can also be included into the maximal independent set and its node(s) is scheduled in the same way as other nodes in interior hexagons.)

V. Probability of Having No Isolated Node

In this section, we obtain a lower bound of P_ω for the scheduling algorithm ω defined in the previous section. This is done by analyzing the event that the network has no isolated node under the scheduling algorithm ω.

We first introduce a lemma, which will be used in the later analysis.

Lemma 2. *Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_6$ denote location vectors of six points placed at the vertices of a hexagon with a unit side length. For an arbitrary point $\boldsymbol{z}$ located inside the hexagon, the following holds: $\sum_{i=1}^{6} \|\boldsymbol{v}_i - \boldsymbol{z}\|^{-\alpha}$ is minimized when $\boldsymbol{z}$ is located at the centre of the hexagon, i.e. $\boldsymbol{z} = \frac{1}{6}\sum_{i=1}^{6} \boldsymbol{v}_i$, where α is the path loss exponent defined in* (3).

Proof: See Appendix A. ■

On the basis of the above lemma, the following theorem, which summarizes a major outcome of the paper, can be obtained:

Theorem 3. *For a CSMA network with n nodes i.i.d. on a $\sqrt{n} \times \sqrt{n}$ square following a uniform distribution. A pair of nodes are directly connected iff both* (1) *and* (2) *($\gamma = 1$ and $N_0 = 0$ in* (1) *and* (2)*) are satisfied. Further, the attenuation function in* (1) *and* (2) *is a power-law function given by* (3) *and the carrier-sensing condition is given by* (4)*. Under the scheduling algorithm ω defined in Section IV, a necessary condition on the transmission power P_ω for the above network to* a.a.s. *have no isolated node as $n \to \infty$ is*

$$P_\omega \geq P_{th} c^\alpha (\log n)^{\frac{\alpha}{2}} \tag{7}$$

where $c = \sqrt{\frac{6}{5}}\,(c'-1)$ and c' is the smallest integer satisfying the inequality:

$$\left(\sqrt{3}c' + \sqrt{3} + 1\right)^2 \geq \frac{5}{2\pi}\,(6\beta)^{\frac{2}{\alpha}} \tag{8}$$

Proof: Let us first consider interior hexagons only. Two hexagons in the same maximal independent set are referred as *neighboring hexagons* if the Euclidean distance between their centers is $L = \sqrt{3}\,(k+1)\,s_n$, which is the Euclidean distance between the centers of two closest hexagons in a maximal independent set (see Figure 1 for an illustration). With this definition, each hexagon has at most six neighboring hexagons. An interior hexagon is called an *inner* hexagon if it has exactly six neighboring interior hexagons in the network area, otherwise it is called an *outer* hexagon. Denote by C_A the area of the union of all inner hexagons. It can be readily shown that $\lim_{n\to\infty} \frac{C_A}{\sqrt{n}\times\sqrt{n}} = 1$.

Let us consider an arbitrarily node i transmitting inside an inner hexagon h_i. If there is no node that can receive from it, then node i is isolated. Let $I_{\min}$ be the minimum interference that could possibly be experienced by a potential receiver of node i under the scheduling algorithm ω. Using (3) and Lemma 2, it can be shown that

$$\begin{aligned} I_{\min} &\geq \sum_{j=1}^{6} P d_j^{-\alpha} \\ &\geq 6P\,(L + s_n)^{-\alpha} \qquad (9) \\ &= 6P\left(\sqrt{3}k + \sqrt{3} + 1\right)^{-\alpha} s_n^{-\alpha} \triangleq J_n \qquad (10) \end{aligned}$$

where d_j represents the Euclidean distance between an active transmitter inside a neighboring hexagon h_j of the hexagon h_i and the receiver of node i. (9) results because the Euclidean distance between the transmitter inside neighboring hexagon h_j and the centre of hexagon h_i is less than $L + s_n$.

Let d be the Euclidean distance between node i and its receiver. Using (1), (2) and (3), it follows that only when $\frac{Pd^{-\alpha}}{J_n} \geq \beta$, the transmission from node i to its receiver could possibly be successful. In other words, if there is no node within an Euclidean distance of $\left(\frac{1}{P}\beta J_n\right)^{-\frac{1}{\alpha}}$ to node i, then node i is isolated. Denote this distance by

$$R = (\beta J_n / P)^{-\frac{1}{\alpha}} \tag{11}$$

Denote by M and M^{SINR} the (random) number of isolated nodes in the CSMA network in the square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ and in $C_A \subset \left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ respectively. Denote by M^{UDM} the (random) number of isolated nodes in an area $C_A \subset \left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ in a network with a total of n nodes i.i.d. and uniformly distributed on the square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ and a pair of nodes are directly connected iff their Euclidean distance is smaller than or equal to R. Based on the discussion in the last paragraph and using the coupling technique [7], it can be shown that $\Pr(M \geq 1) \geq \Pr\left(M^{\text{SINR}} \geq 1\right) \geq \Pr\left(M^{\text{UDM}} \geq 1\right)$. Consequently,

$$\Pr(M = 0) \leq \Pr\left(M^{\text{UDM}} = 0\right) \tag{12}$$

It remains to find the value of $\Pr\left(M^{\text{UDM}}=0\right)$. We first consider a network with a total of n nodes distributed on a square $\left[-\frac{\sqrt{n}}{2},\frac{\sqrt{n}}{2}\right]^2$ and a pair of nodes are directly connected iff their Euclidean distance is smaller than or equal to $r(n)$. It is well-known that when the average node degree in the above network equals to $\log n+\zeta(n)$ and $\lim_{n\to\infty}\zeta(n)=\zeta$ where ζ is a constant ($\zeta=\infty$ is allowed), the probability that there is no isolated node in the above network asymptotically converges to $e^{-e^{-\zeta}}$ as $n\to\infty$[7], [17], [18], [19]. Further, it was shown in [20] that boundary effect has an asymptotically vanishingly impact on the number of isolated nodes. Let $M^{r(n)}$ be the number of isolated nodes within an area C_A in the above network with a transmission range $r(n)$. Using the above results and the fact that $\lim_{n\to\infty}\frac{C_A}{\sqrt{n}\times\sqrt{n}}=1$, it can be concluded that when $r(n)=\sqrt{\frac{\log n+\zeta(n)}{\pi}}$,

$$\lim_{n\to\infty}\Pr\left(M^{r(n)}=0\right)=e^{-e^{-\zeta}} \tag{13}$$

We are now ready to discuss $\Pr\left(M^{\text{UDM}}=0\right)$ and $\Pr(M=0)$. As a result of (12), a necessary condition for $\lim_{n\to\infty}\Pr(M=0)=1$ is that $\lim_{n\to\infty}\Pr\left(M^{\text{UDM}}=0\right)=1$. It follows from (13) that a necessary condition for the network under the SINR model to a.a.s. have no isolated node is that $R\geq\sqrt{\frac{\log n+\zeta(n)}{\pi}}$ and $\zeta(n)\to\infty$ as $n\to\infty$. As a consequence of (11), the above requirement on R means that $\left(\frac{1}{P}\beta J_n\right)^{-\frac{1}{\alpha}}\geq\sqrt{\frac{\log n+\zeta(n)}{\pi}}$ as $n\to\infty$. Substituting the value of J_n in (10) and the value of s_n given in Lemma 1 into the above equation, it can be further obtained that $\left(\sqrt{3}k+\sqrt{3}+1\right)^2\geq\frac{5(\log n+\zeta(n))}{2\pi\log n}(6\beta)^{\frac{2}{\alpha}}$. Letting $n\to\infty$ in the last inequality yields $\left(\sqrt{3}k+\sqrt{3}+1\right)^2\geq\frac{5}{2\pi}(6\beta)^{\frac{2}{\alpha}}$. Based on the above equation, together with (4) and (6), Theorem 3 results. ■

Using Theorem 3 and (5), a necessary condition required for a network with n nodes *i.i.d.* on a $\sqrt{n}\times\sqrt{n}$ square following a uniform distribution to be *a.a.s.* connected under *any* scheduling scheme satisfying carrier-sensing constraint as $n\to\infty$, i.e. a lower bound on P_Ω, is easily obtained:

$$P_\Omega\geq P_\omega\geq P_{\text{th}}c^\alpha(\log n)^{\frac{\alpha}{2}} \tag{14}$$

Comparing the lower bound on P_Ω with the upper bound obtained in [11] (i.e. a sufficient condition for a CSMA network to be *a.a.s.* connected), it can be shown that the lower and upper bounds differs by a constant factor only. Therefore the lower bound given in (14) is reasonably tight.

An implication of the necessary condition is that when the transmission power is set below $P_{\text{th}}c^\alpha(\log n)^{\frac{\alpha}{2}}$, there does not necessarily exist a spatial and temporal scheduling algorithm that allows the network to be connected (in the sense that any pair of nodes can exchange their packets) while in the meantime allowing a maximal set of CSMA transmitters to be active.

VI. Conclusion and Future Work

On the basis of our earlier work [11] in which we give a sufficient condition, i.e. an upper bound, on the transmission power required for a CSMA network with a total of n nodes i.i.d. on a $\sqrt{n}\times\sqrt{n}$ square following a uniform distribution to be *a.a.s.* connected as $n\to\infty$ under the SINR model, in this paper we continued to study the necessary condition for the above CSMA network to be *a.a.s.* connected. A lower bound is obtained on the transmission power required for the above CSMA network to be *a.a.s.* connected under *any* scheduling scheme satisfying carrier-sensing constraint. The lower bound differs from the upper bound by a constant factor only.

Compared with that assuming the unit disk model, it can be easily shown that the critical transmission power for a CSMA network under the SINR model to be *a.a.s.* connected is within a constant factor of that required for the network under the unit disk model to be *a.a.s.* connected. Particularly transmission power only needs to be increased by a constant factor to combat interference and maintain connectivity [11]. This result is much more optimistic than those in [5], [6] considering the connectivity of ALOHA networks under the same SINR model.

It remains our future work to analytically investigate the tightness of the sufficient condition and the necessary condition obtained.

Appendix

In this Appendix, we give a proof of Lemma 2. The proof is based on the following theorem.

Theorem 4. *(Theorem 1 in [21]) Let $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_j$ be j arbitrary points in Euclidean plane. Let $w_1,w_2,\ldots w_j$ be j positive numbers regarded as weights attached to these points, and define a position vector $\boldsymbol{c}$ by*

$$\sum_{i=1}^{j}w_i\boldsymbol{v}_i=W\boldsymbol{c}$$

where

$$W=\sum_{i=1}^{j}w_i$$

Then for an arbitrary point $\boldsymbol{z}$, the following holds:

$$\sum_{i=1}^{j}w_i\|\boldsymbol{v}_i-\boldsymbol{z}\|^2=\sum_{i=1}^{j}w_i\|\boldsymbol{v}_i-\boldsymbol{c}\|^2+W\|\boldsymbol{v}_i-\boldsymbol{c}\|^2$$

Now we are ready to prove Lemma 2.

Consider the six points defined in Lemma 2. Letting all attached weights w_i equal to 1 and using Theorem 4, it follows that

$$\textstyle\sum_{i=1}^{6}\|\boldsymbol{v}_i-\boldsymbol{z}\|^2=\sum_{i=1}^{6}\|\boldsymbol{v}_i-\boldsymbol{c}\|^2+6\|\boldsymbol{z}-\boldsymbol{c}\|^2, \tag{15}$$

where $\boldsymbol{c}$ is given by

$$\sum_{i=1}^{6}\boldsymbol{v}_i=6\boldsymbol{c}. \tag{16}$$

It is clear that $\boldsymbol{c}$ is the centroid of the six points. Since the hexagon has a unit side length, $\|\boldsymbol{v}_i-\boldsymbol{c}\|$ equals to 1. Let $x_i=$

$\|\boldsymbol{v}_i - \boldsymbol{z}\|$ and $y = \|\boldsymbol{z} - \boldsymbol{c}\|$. Lemma 2 can be converted to the following constrained minimization problem:

$$\begin{aligned} \text{minimize} \quad & f(x_1, x_2, \ldots, x_6) = \textstyle\sum_{i=1}^{6} x_i^{-\alpha} \\ \text{subject to} \quad & h(x_1, x_2, \ldots, x_j) \\ = \quad & \textstyle\sum_{i=1}^{6} x_i^2 - 6 - 6y^2 = 0 \end{aligned}$$

where the constraint is due to (15). Using the method of Lagrange multipliers, we first construct the Lagrangian in the following:

$$F(x_1, x_2, \ldots x_6, \lambda) = f(x_1, x_2, \ldots, x_6) + \lambda h(x_1, x_2, \ldots, x_6)$$

where the parameter λ is known as the Lagrange multiplier. Then find the gradient and set it to zero:

$$\nabla F(x_1, x_2, \ldots x_6, \lambda) = \begin{pmatrix} -\alpha x_1^{-\alpha-1} + 2\lambda x_1 \\ -\alpha x_2^{-\alpha-1} + 2\lambda x_2 \\ \ldots \\ -\alpha x_6^{-\alpha-1} + 2\lambda x_6 \\ h(x_1, x_2, \ldots, x_6) \end{pmatrix}^T = \mathbf{0}.$$

Solving the above equation, it is obtained that

$$\lambda = \frac{\alpha}{2}\left(1 + y^2\right)^{-\frac{\alpha+2}{2}}$$

$$\text{and } x_1 = x_2 = \ldots = x_6 = \left(\frac{2\lambda}{\alpha}\right)^{\frac{-1}{\alpha+2}} = \left(1 + y^2\right)^{\frac{1}{2}}.$$

Since $x_i = \|\boldsymbol{v}_i - \boldsymbol{z}\|$ denotes the Euclidean distance from $\boldsymbol{v}_i$ to $\boldsymbol{z}$, only when $\boldsymbol{z} = \boldsymbol{c}$, we can have $x_1 = x_2 = \ldots = x_6 = 1$. It follows that the minimum of $f(x_1, x_2, \ldots, x_6)$ is obtained only when $\boldsymbol{z}$ is located at the centroid of the hexagon. Hence, the result follows.

References

[1] P. Gupta and P. R. Kumar, "Critical power for asymptotic connectivity," in *Proc. IEEE Conference on Decision and Control*, 1998, pp. 1106–1110.

[2] M. Haenggi, J. G. Andrews, F. Baccelli, O. Dousse, and M. Franceschetti, "Stochastic geometry and random graphs for the analysis and design of wireless networks," *IEEE Journal on Selected Areas in Communications*, vol. 27, no. 7, pp. 1029–1046, 2009.

[3] C. Bettstetter and C. Hartmann, "Connectivity of wireless multihop networks in a shadow fading environment," *Wireless Networks*, vol. 11, no. 5, pp. 571–579, 2005.

[4] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Transactions on Information Theory*, vol. 46, no. 2, pp. 388–404, 2000.

[5] O. Dousse, F. Baccelli, and P. Thiran, "Impact of interferences on connectivity in ad hoc networks," *IEEE/ACM Transactions on Networking*, vol. 13, no. 2, pp. 425–436, 2005.

[6] O. Dousse, M. Franceschetti, N. Macris, R. Meester, and P. Thiran, "Percolation in the signal to interference ratio graph," *Journal of Applied Probability*, vol. 43, no. 2, pp. 552–562, 2006.

[7] M. Franceschetti and R. Meester, *Random Networks for Communication from Statistical Physics to Information Systems*. Cambridge University Press, 2007.

[8] S. Kumar, V. S. Raghavan, and J. Deng, "Medium access control protocols for ad hoc wireless networks: A survey," *Ad Hoc Networks*, vol. 4, no. 3, pp. 326–358, 2006.

[9] R. Gallager, "A perspective on multiaccess channels," *IEEE Transactions on Information Theory*, vol. 31, no. 2, pp. 124–142, 1985.

[10] T. S. Rappaport, *Wireless Communications Principles and Practice*, 2nd ed. Prentice Hall, 2002.

[11] T. Yang, G. Mao, and W. Zhang, "Connectivity of wireless csma multi-hop networks," in *Proc. IEEE International Conference on Communications, 2011*, pp. 1–5.

[12] M. Penrose, *Random Geometric Graphs*, 1st ed. New York: Oxford University Press, 2003.

[13] R. Hekmat and P. Van Mieghem, "Connectivity in wireless ad-hoc networks with a log-normal radio model," *Mobile Networks & Applications*, vol. 11, no. 3, pp. 351–360, 2006.

[14] C. Avin, Z. Lotker, F. Pasquale, and Y. A. Pignolet, "A note on uniform power connectivity in the sinr model," *Algorithmic Aspects of Wireless Sensor Networks*, vol. 5804, pp. 116–127, 2009.

[15] E. Lebhar and Z. Lotker, "Unit disk graph and physical interference model: Putting pieces together," in *IEEE International Symposium on Parallel&Distributed Processing*, 2009, pp. 1–8.

[16] M. Haenggi and R. K. Ganti, "Interference in large wireless networks," *Found. Trends Netw.*, vol. 3, no. 2, pp. 127–248, 2009.

[17] C.-W. Yi, P.-J. Wan, X.-Y. Li, and O. Frieder, "Asymptotic distribution of the number of isolated nodes in wireless ad hoc networks with bernoulli nodes," *IEEE Transactions on Communications*, vol. 54, no. 3, pp. 510–517, 2006.

[18] M. Franceschetti and R. Meester, "Critical node lifetimes in random networks via the chen-stein method," *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2831–2837, 2006.

[19] G. Mao and B. D. O. Anderson, "On the asymptotic connectivity of random networks under the random connection model," in *Proc. IEEE INFOCOM*, 2011, pp. 631–639.

[20] G. Mao and B. D. Anderson, "Towards a better understanding of large scale network models," *accepted to appear in IEEE/ACM Transactions on Networking, available at http://arxiv.org/abs/1012.5723*, 2011.

[21] T. M. Apostol and M. A. Mnatsakanian, "Sums of squares of distances in m-space," *American Mathematical Monthly*, vol. 110, no. 6, pp. 516–526, 2003.