

# Capacity of Interference-limited Three Dimensional CSMA Networks

Tao Yang, Guoqiang Mao

School of Electrical and Information Engineering  
The University of Sydney

Email: {tao.yang, guoqiang.mao}@sydney.edu.au

Wei Zhang

School of Electrical Engineering and Telecommunications  
The University of New South Wales

Email: wzhang@ee.unsw.edu.au

**Abstract**—In this paper, we study the throughput of interference-limited three dimensional (3D) CSMA networks. Specifically, we consider a network with a total of  $n$  nodes uniformly i.i.d. in a cube of edge length  $n^{\frac{1}{3}}$ . Further, CSMA random access scheme is employed and the SINR model is used to simulate a successful transmission. We first give a sufficient condition on the transmit power required for the CSMA network to be asymptotically almost surely (a.a.s.) connected as  $n \rightarrow \infty$  under the SINR model. Then, we demonstrate constructively that a throughput of  $\Theta\left(\frac{1}{(n \log^2 n)^{\frac{1}{3}}}\right)$  is obtainable by each node for an arbitrarily chosen destination.

## I. INTRODUCTION

Wireless multi-hop networks have been increasingly used in military and civilian applications. In many applications, the region in which the network is deployed is better modeled by a 3D space, instead of a two dimensional (2D) planar area. Examples include a wireless network deployed across different floors inside a building connecting a variety of devices such as computers, smart phones, sensors etc, a network formed by Unmanned Aerial Vehicles and ground devices for reconnaissance and surveillance, and underwater acoustic sensor networks. Capacity of such networks is an important problem. The scaling behavior of capacity when the network becomes sufficiently large is of particular interest.

Existing work on the capacity of wireless multi-hop networks has mainly focused on the analysis of 2D networks [1], [2], including [1] which considered 2D CSMA networks. Limited work has considered the properties of 3D networks where centralized/deterministic scheduling schemes like TDMA are employed [3], [4]. On the other hand, CSMA schemes, which make use of *distributed/randomized* medium access protocols, has become prevailing with widespread adoption. With CSMA, each node checks the status of the wireless channel before sending a packet. If the channel is idle (i.e. no carrier is detected within its carrier-sensing range), then the node starts its transmission, otherwise, defers it, usually by a random amount of time, until the channel becomes idle again. Potential transmitters in the vicinity of an active transmitter are kept

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off. Wireless signals transmitted at the same time mutually interfere with each other. The SINR (signal to interference plus noise ratio) model has been widely used to capture the impact of interference on the quality of a link and a transmission is considered to be successful iff a minimum SINR requirement has been met [2]. Therefore, it is natural to expect CSMA could improve the network performance by alleviating the interference.

In this paper, we consider a CSMA network with  $n$  nodes uniformly i.i.d. in a cube of edge length  $n^{\frac{1}{3}}$  and investigate the throughput of the network. The contributions of this paper are:

- 1) We derive an upper bound on the interference experienced by any receiver in the 3D CSMA network. Using the result, we show that for an arbitrary SINR requirement, there exists a transmission range  $R_0$  such that any two nodes are directly connected if their Euclidean distance is less than or equal to  $R_0$ . Based on that, we give a sufficient condition on the transmit power for the CSMA network to be a.a.s. connected under the SINR model as  $n \rightarrow \infty$ . A connected network is a prerequisite for the network to achieve a non-zero throughput.
- 2) We further show that in the 3D CSMA network, a throughput of  $\Theta\left(\frac{1}{(n \log^2 n)^{\frac{1}{3}}}\right)$  is achievable. Compared with the results in [3] and [4], which showed that a throughput of  $\Theta\left(\frac{1}{(n \log^2 n)^{\frac{1}{3}}}\right)$  is attainable by using either a deterministic scheduling [3], [4] or without considering the SINR requirement for successful transmissions [4], our result shows that a throughput of  $\Theta\left(\frac{1}{(n \log^2 n)^{\frac{1}{3}}}\right)$  is also attainable even when CSMA is used and a minimum SINR is required.

The remainder of this paper is organized as follows: Section II reviews related work; Section III defines the network and metrics being investigated; in Section IV, we give a sufficient condition on the transmit power to have an a.a.s. connected 3D CSMA network; In Section V, we obtain a lower bound on the achievable throughput of 3D CSMA networks; Section VI concludes the paper and discusses future work.

## II. RELATED WORK

Existing work on studying capacity problem focused mainly on 2D networks. The seminal work [2] showed that in a network with a total of  $n$  nodes distributed on a disk of

unit area under the SINR model, the throughput obtainable by each node is 1)  $\Theta\left(\frac{1}{\sqrt{n \log n}}\right)$  if nodes are randomly i.i.d. and destination is randomly chosen for each node; 2)  $\Theta\left(\frac{1}{\sqrt{n}}\right)$  if nodes' location, traffic pattern and transmission range are optimally arranged. Since this pioneering work, extensive efforts have been made to investigate the capacity in different scenarios. Significant outcomes have been achieved for both static networks [5]–[7] and mobile networks [8], [9]. The paper [9] showed that mobility of nodes can be exploited to significantly improve network capacity at the expense of delay. Other work in the area includes [10], [11] studied the capacity of networks with infrastructure support, and [11] showed that randomly placed base stations can also boost the throughput.

For networks using distributed/random CSMA scheme, the recent work [1] showed that a throughput of  $\Theta\left(\frac{1}{\sqrt{n}}\right)$  can be achieved in 2D CSMA networks with randomly chosen destinations. The result is in the same order as the TDMA network considered in [2]. [12], [13] studied the interactions between the transmit power, the carrier-sensing range and the capacity in 2D CSMA networks.

Very limited work (see [3], [4] and references therein) has studied the capacity of 3D networks and all focused on networks employing deterministic schedulings. The paper [3] considered a network deployed in a sphere and showed that a throughput of  $\Theta\left(\frac{1}{(n \log^2 n)^{\frac{1}{3}}}\right)$  is feasible. A more recent work [4] studied the capacity of 3D networks under two scenarios, i.e. nodes are regularly placed and nodes are Poissonly distributed.

### III. NETWORK MODELS AND PRELIMINARIES

We consider a network with  $n$  nodes uniformly i.i.d. in a cube with edge length  $n^{\frac{1}{3}}$ .

#### A. Interference model

Assume all nodes use a common transmit power  $P$ . Let  $\mathbf{x}_k$ ,  $k \in \Gamma$ , be the location of node  $k$ , where  $\Gamma$  represents the set of indices of all nodes in the network. A transmission from node  $i$  to node  $j$  is successful iff the SINR at node  $j$  is above a threshold  $\beta$ , i.e.

$$\text{SINR}(\mathbf{x}_i, \mathbf{x}_j) = \frac{P\ell(\mathbf{x}_i, \mathbf{x}_j)}{N_0 + \sum_{k \in \mathcal{T}_i} P\ell(\mathbf{x}_k, \mathbf{x}_j)} \geq \beta \quad (1)$$

where  $\mathcal{T}_i \subseteq \Gamma$  denotes the subset of nodes transmitting at the same time as node  $i$ .  $\ell(\mathbf{x}_i, \mathbf{x}_j)$  represents power attenuation from  $\mathbf{x}_i$  to  $\mathbf{x}_j$  and assumes a power-law form, i.e.,

$$\ell(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|^{-\alpha} \quad (2)$$

where  $\|\cdot\|$  is the Euclidean norm and  $\alpha$  is the path-loss exponent. We assume that the background noise  $N_0$  is negligibly small, i.e.  $N_0 = 0$ . This assumption is justified because interference is a major factor that weakens performance in wireless networks, while the background noise is typically small and can be combated by increasing the transmit power. As commonly done in the capacity analysis [2]–[4], [9], [11], the impact of small-scale fading is ignored.

Since CSMA typically require an ACK packet to acknowledge a successful transmission, we explicitly consider *bidirectional link* only in the network. In other words, a transmission from node  $i$  to node  $j$  is successful iff both SINR  $(\mathbf{x}_i, \mathbf{x}_j)$  and SINR  $(\mathbf{x}_j, \mathbf{x}_i)$  are above  $\beta$ . In that case, we also say that node  $i$  and  $j$  are directly connected.

#### B. Definition of throughput

The *channel rate* of a transmission from node  $i$  to node  $j$  is related to the associated SINR by Shannon theorem, i.e.,

$$R(\mathbf{x}_i, \mathbf{x}_j) = B \log_2(1 + \text{SINR}(\mathbf{x}_i, \mathbf{x}_j)) \quad (3)$$

where  $B$  is the bandwidth of the channel in Hertz. Due to the minimum SINR requirement in (1), the channel rate between a pair of directly connected nodes is at least  $B \log_2(1 + \beta)$ .

Every node sends data at a rate (bits/sec) to a randomly chosen destination. A node is both a source and a destination node for another node. Therefore the total number of source-destination pairs is  $n$ . The per-node *throughput*, denoted by  $\lambda(n)$ , is defined as the *maximum* rate that could be achieved by any source-destination pair simultaneously. A throughput of  $\lambda(n)$  is *feasible* if there is a temporal and spatial scheduling scheme such that every node can send  $\lambda(n)$  bits/sec *on average* to its destination, i.e. there exists a sufficiently large positive number  $\tau$  such that in every finite time interval  $[(j-1)\tau, j\tau]$  every node can send  $\tau\lambda(n)$  bits to its destination. A throughput is of order  $\Theta(f(n))$  bits/sec if there are deterministic constants  $0 < c < c' < +\infty$  such that  $\lim_{n \rightarrow \infty} \Pr(\lambda(n) = cf(n) \text{ is feasible}) = 1$  and  $\lim_{n \rightarrow \infty} \Pr(\lambda(n) = c'f(n) \text{ is feasible}) < 1$ .

#### C. CSMA random access scheme

In CSMA networks, two nodes, say  $i$  and  $j$ , are allowed to transmit simultaneously if they can not detect each other's transmission, i.e. both  $P\ell(\mathbf{x}_i, \mathbf{x}_j)$  and  $P\ell(\mathbf{x}_j, \mathbf{x}_i)$  are under a certain detection threshold  $P_{\text{th}}$  (this is also termed as the *pairwise carrier-sensing decision model* in [1]). This mechanism imposes a *minimum* separation constraint among the concurrent transmitters, known as the *carrier-sensing range*. It readily follows from (2) that the carrier-sensing range  $R_c$  is given by

$$R_c = (P/P_{\text{th}})^{1/\alpha} \quad (4)$$

Under the carrier-sensing constraint, multiple nodes contend for an opportunity to transmit and at a particular time instant, there can only be one node transmitting in a geographic region determined by  $R_c$ . Therefore, the channel rate given by (3) is shared by several nodes in the vicinity over time. Next, we describe how to obtain the *time-average* channel rate (or equivalently the *long-term* channel rate in [1]) for each node.

Same as that in [1], we consider an idealized CSMA scheme. Assume that each node maintains a *countdown timer*, which is initialized to a non-negative random integer. The timer of a node counts down when the node senses the channel idle, otherwise it is frozen. A node initiates its transmission when its timer reaches zero *and* the channel is idle. After finishing transmission, the node resets its timer to a new

random integer. The average countdown time can be distinct for different nodes, which can be set to control the state transition probabilities in the next paragraph.

The above CSMA scheme can be modeled by a Markov chain with state space  $\mathbb{S}$ , where a state  $\mathcal{S} \in \mathbb{S}$  represents the active transmitter set at a particular time instant. A transition between two distinct states  $\mathcal{S}, \mathcal{S}' \in \mathbb{S}$  can possibly occur iff  $\mathcal{S}' = \{i\} \cup \mathcal{S}$  for  $\forall i \in \Gamma$ . Transition  $\mathcal{S} \rightarrow \{i\} \cup \mathcal{S}$  (where  $i \notin \mathcal{S}$ ) represents the event that node  $i$  will start its transmission after its timer counts down to zero. Transition  $\{i\} \cup \mathcal{S} \rightarrow \mathcal{S}$  represents the event that node  $i$  finishes its transmission and hence become silent again. Let  $v$  be the set of state transition probabilities and denote the above Markov chain by  $\langle \mathbb{S}, v \rangle$ . Then the time-average channel rate available for each node can be characterized by the stationary distribution of  $\langle \mathbb{S}, v \rangle$  [14] [1, Lemma 8].

#### IV. CONNECTIVITY OF 3D CSMA NETWORKS

For any throughput to be feasible, a prerequisite is that there exists a path between each pair of source and destination, i.e. the network is connected. In this section, we first derive an upper bound on the interference experienced by any receiver in the network. We further show that for an arbitrarily chosen  $\beta$ , there exists a transmission range  $R_0$  such that a pair of nodes are directly connected if their Euclidean distance is smaller than or equal to  $R_0$ . Based on that, we give a sufficient condition on the transmit power for the 3D CSMA network to be a.a.s. connected as  $n \rightarrow \infty$  under the SINR model.

The following lemma gives an upper bound on the interference.

**Lemma 1.** *Consider a CSMA network with nodes arbitrarily distributed in a region in  $\mathfrak{R}^3$  where the carrier-sensing range is  $R_c$ , given by (4). Denote by  $r_0$  the Euclidean distance between an arbitrary receiver and its intended transmitter and  $r_0 < R_c$ . When the path loss exponent  $\alpha > 3$ , the maximum interference is upper bounded by  $N(r_0)$ , where*

$$N(r_0) = 12P(R_c - r_0)^{-\alpha} + 17P \left( \frac{\sqrt{6}}{3} R_c - r_0 \right)^{-\alpha} + \frac{9P \left( \left( \frac{251}{6} \alpha^2 - \frac{769}{6} \alpha + 89 \right) R_c^2 - 27\sqrt{6} R_c r_0 (\alpha - 1) + 54r_0^2 \right)}{2\sqrt{6} (\alpha - 1) (\alpha - 2) (\alpha - 3) R_c^3 \left( \frac{\sqrt{6}}{2} R_c - r_0 \right)^{\alpha - 1}} \quad (5)$$

*Proof:* See Appendix. ■

**Remark 2.** Note that the upper bound in Lemma 1 applies to arbitrary node distribution in  $\mathfrak{R}^3$ . Further, the requirement that  $\alpha > 3$  is for the interference to be bounded by a constant independent of  $n$ . If  $\alpha \leq 3$ , then the interference given by (13) approaches infinity as  $n \rightarrow \infty$ . In that case, an upper bound on interference can still be found by using the technique presented in the proof of Lemma 1 but that bound will be a function of  $n$ . In this paper, we focus on the situation that  $\alpha > 3$  to avoid some verbose but straightforward discussion on special cases that occur when  $\alpha \leq 3$ .

Corollary 3 is a consequence of Lemma 1.

**Corollary 3.** *Under the same setting as that in Lemma 1, there exists a transmission range  $R_0 < R_c$  such that a pair*

*of nodes are directly connected if their Euclidean distance is smaller than or equal to  $R_0$ , which is given implicitly by*

$$PR_0^{-\alpha} / N(R_0) = \beta \quad (6)$$

*Proof:* Noting that  $\frac{Pr_0^{-\alpha}}{N(r_0)} \rightarrow \infty$  as  $r_0 \rightarrow 0$ ,  $\frac{Pr_0^{-\alpha}}{N(r_0)} \rightarrow 0$  as  $r_0 \rightarrow R_c^-$  and that  $\frac{Pr_0^{-\alpha}}{N(r_0)}$  is a monotonically decreasing function of  $r_0$ , therefore there is a unique solution to (6). The rest of the proof is trivial and hence omitted. ■

Since  $P = P_{th} R_c^\alpha$ ,  $R_0$  in (6) can also be expressed as a function of  $R_c$ . Letting  $\frac{R_c}{R_0} = x$ , (6) can be rewritten as

$$\frac{1}{\beta} = 12(x-1)^{-\alpha} + 17 \left( \frac{\sqrt{6}}{3} x - 1 \right)^{-\alpha} + \frac{9 \left( \left( \frac{251}{6} \alpha^2 - \frac{769}{6} \alpha + 89 \right) x^2 - 27\sqrt{6} (\alpha - 1) x + 54 \right)}{2\sqrt{6} (\alpha - 1) (\alpha - 2) (\alpha - 3) x^3 \left( \frac{\sqrt{6}}{2} x - 1 \right)^{\alpha - 1}} \quad (7)$$

It follows that  $R_0 = \frac{R_c}{b}$  and  $b$  is the solution to (7), which depends on  $\beta$  and  $\alpha$  only. Equation (7) gives a more convenient way to study the relation between  $P$  and  $R_0$ .

Based on Corollary 3 and the result in [15, Theorem 1] on the connectivity of 3D networks under the unit disk connection model, we obtain the following theorem.

**Theorem 4.** *Consider a CSMA network with a total of  $n$  nodes uniformly i.i.d. in a cube with edge length  $n^{\frac{1}{3}}$ . Under SINR model, the network is a.a.s. connected as  $n \rightarrow \infty$  if transmit power*

$$P = P_{th} (b')^\alpha (\log n + c(n))^{\frac{\alpha}{3}} \quad (8)$$

where  $\lim_{n \rightarrow \infty} c(n) = +\infty$ ,  $b' = b(3/4\pi)^{\frac{1}{3}}$  and  $\infty > b > 1$  is the solution to (7).

*Proof:* The theorem readily follows from the result in [15, Theorem 1] with proper scaling and Corollary 3. ■

An implication of Theorem 4 is that when  $P$  is set as that in (8), a.a.s. there exists a temporal and spatial scheduling scheme that allow any pair of nodes in the CSMA network to exchange packets.

#### V. FEASIBLE THROUGHPUT

In this section, we first describing a routing algorithm and then show that a throughput of  $\Theta \left( \frac{1}{(n \log^2 n)^{\frac{1}{3}}} \right)$  can be achieved using the routing algorithm in 3D CSMA networks.

Partition the cube into non-overlapping cubelets of edge length  $s_n = (4 \log n)^{\frac{1}{3}}$ . Let  $X_i$  be the random number of nodes in a cubelet  $i$ . Let  $\bar{X} = \max_{i \in \mathbb{T}} X_i$  and  $\underline{X} = \min_{i \in \mathbb{T}} X_i$  where  $\mathbb{T}$  represents the set of indices of all cubelets. We obtain:

**Lemma 5.** *As  $n \rightarrow \infty$ ,  $\Pr(\bar{X} \leq c_1 \log n) = 1$  and  $\Pr(\underline{X} \geq c_2 \log n) = 1$  where  $c_1 = 4 \left( 1 + \frac{\sqrt{3}}{2} \right)$  and  $c_2 = 4 \left( 1 - \frac{\sqrt{2}}{2} \right)$ .*

*Proof:* Note that  $X_i$  has a binomial distribution with parameters  $n$  and  $\frac{s_n^3}{n}$  and  $E[X_i] = s_n^3 = 4 \log n$ . Using the Chernoff bound, we have that for any  $\delta \in (0, 1)$ ,  $\Pr[X_i \geq (1 + \delta) E[X_i]] \leq \exp \left( -\frac{\delta^2 E[X_i]}{3} \right)$  holds.

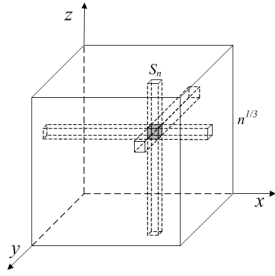


Figure 1. An illustration on the number of routes served by one cubelet.

Let  $\delta = \frac{\sqrt{3}}{2}$ , then  $\Pr \left[ X_i \geq 4 \left( 1 + \frac{\sqrt{3}}{2} \right) \log n \right] \leq \frac{1}{n}$ . There are a total of  $\frac{n}{s_n^3}$  cubelets (here we ignored some trivial discussion on granularity problem caused by  $\frac{n}{s_n^3}$  not being an integer). By the union bound, we have  $\lim_{n \rightarrow \infty} \Pr \left[ \bigcup_{i=1}^{n/s_n^3} \left( X_i \geq 4 \left( 1 + \frac{\sqrt{3}}{2} \right) \log n \right) \right] = 0$ .

Using a similar method, we have that for any  $\delta' \in (0, 1)$   $\Pr [X \leq (1 - \delta') E[X]] \leq \exp \left( -\frac{\delta'^2 E[X]}{2} \right)$  holds. Taking  $\delta' = \frac{\sqrt{2}}{2}$  and using the union bound yields  $\underline{X}$ . ■

#### A. The maximum traffic served by each cubelet

For a given  $\beta$ , using (7), (4), we can choose a transmit power so that the transmission range  $R_0$ , given by Corollary 3, is  $\sqrt{6}s_n$ . This value allows any two nodes in two neighboring cubelets to directly communicate with each other under the SINR model. Hence, the channel rate between two nodes, whose Euclidean distance is less than or equal to  $R_0$ , is at least  $B \log_2(1 + \beta)$ . Using (7) we can write:  $R_c = b\sqrt{6}s_n$ , where  $b$  is the solution to (7) for the given  $\beta$ .

We employ a similar routing scheme to that used in [4]. For a pair of source and destination nodes located at  $(x_s, y_s, z_s)$  and  $(x_d, y_d, z_d)$  respectively, packets generated by the source are first relayed to a node closest to  $(x_s, y_s, z_d)$ , then to a node closest to  $(x_d, y_s, z_d)$  and finally delivered to  $(x_d, y_d, z_d)$ .

As illustrated in Fig. 1, the shaded space represents a cubelet. According to the above routing scheme, only nodes located in the three rectangular cuboid, which are bounded by dashed lines in the figure, possibly need nodes in the shaded cubelet to relay their data. Therefore the maximum number of routes served by each cubelet is

$$N_{\text{routes}} = 3 \frac{n^{1/3}}{s_n} \bar{X} = \frac{3\bar{X}}{4^{1/3}} \left( \frac{n}{\log n} \right)^{1/3} \quad (9)$$

Suppose each node sends data at a rate  $\lambda(n)$  bits/sec to its destination. The maximum amount of traffic each cubelet needs to transmit is  $N_{\text{routes}} \lambda(n)$  at most.

#### B. Time-average channel rate for each node

In this subsection, we first construct a deterministic TDMA scheduling  $(\mathcal{S}_t)_{t=1}^m$ , where  $\mathcal{S}_t \in \mathbb{S}$  is the active transmitter set during time slot  $t$ , and determine the time-average channel rate  $C_i^{\text{det}}[(\mathcal{S}_t)_{t=1}^m]$  for node  $i \in \Gamma$  under this scheme. Then using the result in [1, Lemma 9], we show that  $C_i^{\text{det}}[(\mathcal{S}_t)_{t=1}^m] \leq C_i^{\text{ran}}[\langle \mathbb{S}, v \rangle]$  where  $C_i^{\text{ran}}[\langle \mathbb{S}, v \rangle]$  is the time-average channel rate for node  $i$  under the CSMA scheme described in Section

III-C (modeled by the Markov chain  $\langle \mathbb{S}, v \rangle$ ). Finally we establish a lower bound on  $C_i^{\text{ran}}[\langle \mathbb{S}, v \rangle]$ .

As shown in (3), any pair of directly connected nodes can transmit at a rate at least  $B \log_2(1 + \beta)$ . For convenience, in the following discussion, we consider that the rate equals to  $B \log_2(1 + \beta)$  and normalize it to 1. We divide time into slots of unit length. It follows that the channel rate available for a particular node  $i$  under the scheduling scheme  $(\mathcal{S}_t)_{t=1}^m$  is equal to the fraction of time that node  $i$  gets to transmit, i.e.

$$C_i^{\text{det}}[(\mathcal{S}_t)_{t=1}^m] = \frac{1}{m} \sum_{t=1}^m \mathbf{1}(i \in \mathcal{S}_t)$$

We group adjacent cubelets into non-overlapping cubes and each cube contains  $(k+1)^3$  cubelets, where  $k = \lceil b\sqrt{6} \rceil$  so that  $ks_n \geq R_c = b\sqrt{6}s_n$ . Using Lemma 5, a.s. there are at most  $c_1 \log n$  nodes in every cubelet. Based on the above discussion, a deterministic scheduling algorithm can be designed such that within time slots from  $t = 1$  to  $t = (k+1)^3 c_1 \log n$ , each node gets at least one time slot to transmit while the set of concurrent transmitters meets the CSMA constraints. Denote by  $\mathcal{S}_t$  the concurrent transmitter set during time slot  $t$ . It follows that  $\mathcal{S}_t \in \mathbb{S}$  for  $t \in [1, (k+1)^3 c_1 \log n]$  and the fraction of time spent on each  $\mathcal{S}_t$ ,  $t \in [1, (k+1)^3 c_1 \log n]$  is  $\frac{1}{(k+1)^3 c_1 \log n}$ . Letting  $m = (k+1)^3 c_1 \log n$ , it then follows that there is a deterministic scheduling that can achieve a time-average channel rate of at least  $\frac{B \log_2(1+\beta)}{(k+1)^3 c_1 \log n}$  for node  $i$ , i.e.

$$C_i^{\text{det}}[(\mathcal{S}_t)_{t=1}^m] \geq \frac{B \log_2(1 + \beta)}{(k+1)^3 c_1 \log n} \quad (10)$$

Using the result in [1, Lemma 9] which states that there exists a properly designed CSMA scheme that delivers suitable state transition probabilities  $v$ , such that for each node  $i$ , the following holds

$$C_i^{\text{ran}}[\langle \mathbb{S}, v \rangle] \geq C_i^{\text{det}}[(\mathcal{S}_t)_{t=1}^m] \quad (11)$$

where  $C_i^{\text{det}}[(\mathcal{S}_t)_{t=1}^m]$  in (11) is the time-average channel rate available for node  $i$  under a deterministic scheduling scheme.

Combining (11), (10) and (3), it can be established that for node  $i \in \Gamma$ ,  $C_i^{\text{ran}}[\langle \mathbb{S}, v \rangle] \geq \frac{B \log_2(1+\beta)}{(k+1)^3 c_1 \log n}$ .

Lemma 5 also tells that the minimum number of nodes in every cubelet is greater than or equal to  $c_2 \log n$  a.s. Therefore, the minimum time-average channel rate available for each cubelet under CSMA scheme is

$$\frac{(c_2 \log n) B \log_2(1 + \beta)}{(k+1)^3 c_1 \log n} = \frac{c_2 B \log_2(1 + \beta)}{c_1 (k+1)^3} \quad (12)$$

#### C. Lower bound on throughput

For any per-node throughput  $\lambda(n)$  to be feasible, the traffic load for each cubelet should not exceed the time-average channel rate available for each cubelet, i.e.,

$$N_{\text{routes}} \lambda(n) \leq \frac{c_2 B \log_2(1 + \beta)}{c_1 (k+1)^3} \text{ bits/sec}$$

which results in a lower bound on the feasible per-node throughput. This is summarized in the Theorem 6, which forms another major contribution of this paper.

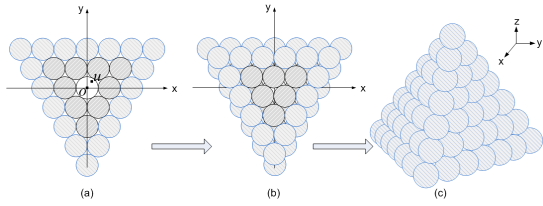


Figure 2. Densest sphere packing.

**Theorem 6.** For the considered CSMA networks, there exists a deterministic constant  $c > 0$ , independent of  $\alpha$ ,  $P_{th}$  and  $\beta$ , such that a per-node throughput  $\lambda(n) = \frac{c \log_2(1+\beta)}{(k+1)^3} \left( \frac{1}{(n \log^2 n)^{\frac{1}{3}}} \right)$  bits/sec is feasible a.a.s. as  $n \rightarrow \infty$ , where  $k = \lceil b\sqrt{6} \rceil$  and  $b$  is the solution to (7) for a given  $\beta$ .

## VI. CONCLUSION

In this paper, we studied the throughput of 3D CSMA networks. We first provided a sufficient condition on the transmit power for having an a.a.s. connected 3D CSMA network under the SINR model. Then, using a simple routing scheme, we obtained a per-node throughput of  $\Theta\left(\frac{1}{(n \log^2 n)^{\frac{1}{3}}}\right)$  is feasible even when distributed/random access scheme is used and a minimum SINR for each successful transmission is specified. It remains our future work to study the optimum access scheme that maximizes the per-node throughput.

## APPENDIX: PROOF OF LEMMA 1

The derivation of the upper bound on interference is similar to that used in [16]. The difference is in that, here we use densest sphere packing in 3D space to derive the upper bound.

Construct a coordinate system such that the origin  $o$  is at a transmitter  $w$ . Consider a transmission from  $w$  to its receiver  $u$  located at  $\mathbf{u}$  and define  $r_0 \triangleq \|\mathbf{u}\|$ .

Draw a sphere of radius  $R_c/2$  centered at each concurrent transmitter. The two spheres centered at two closest transmitters cannot overlap. The maximum interference happens when these spheres are placed in the densest way, which is to place the sphere centers at the vertices of a *face-centered cubic lattice* [17, p.9]. See Fig. 2 for an illustration.

Group the sphere centers into tiers of increasing distance from the origin. The picture (a) in Fig.2 shows the spheres in the 1<sup>st</sup> tier (in dark shade) and the 2<sup>nd</sup> tier (in light shade) whose centers are located on the  $x - y$  plane. All of the spheres in the 2<sup>nd</sup> tier whose centers are above the  $x - y$  plane (including those on the  $x - y$  plane) are shown in light shade in picture (c). Although Fig.2 only shows the packing along  $+z$  axis, the packing goes along  $-z$  axis as well in the same way as along  $+z$  axis. The number of interferers in the  $j^{\text{th}}$  tier is  $27j^2 + 2$ . Let  $\mathbf{x}_i$  be the location of an interferer. The minimum distance from an interferer in the  $j^{\text{th}}$  tier to the origin is  $\frac{\sqrt{6}}{3}jR_c$ . Denote by  $I(r_0)$  the interference at node  $z$ . Since  $\|\mathbf{x}_i - \mathbf{u}\| \geq \|\mathbf{x}_i\| - r_0$ , (2) we obtain

$$I(r_0) \leq \frac{12P}{(R_c - r_0)^\alpha} + \frac{17P}{\left(\frac{\sqrt{6}}{3}R_c - r_0\right)^\alpha} + \sum_{j=2}^{\infty} \frac{P(27j^2 + 2)}{\left(\frac{\sqrt{6}}{3}jR_c - r_0\right)^\alpha} \quad (13)$$

The first two items in RHS of (13) account for the interference caused by the 1<sup>st</sup> tier interferers. Let  $U_j$ ,  $j = 2, \dots, \infty$ , be random variables uniformly and i.i.d. in  $\left[j - \frac{1}{2}, j + \frac{1}{2}\right]$ . It follows from the convexity of  $(27j^2 + 2) \left(\frac{\sqrt{6}}{3}jR_c - r_0\right)^{-\alpha}$  when  $j \geq 2$ ,  $\alpha > 3$  and Jensen's inequality that

$$\begin{aligned} & P \sum_{j=2}^{\infty} (27j^2 + 2) \left(\frac{\sqrt{6}}{3}jR_c - r_0\right)^{-\alpha} \\ &= P \sum_{j=2}^{\infty} \left(27\mathbb{E}(U_j)^2 + 2\right) \left(\frac{\sqrt{6}}{3}\mathbb{E}(U_j)R_c - r_0\right)^{-\alpha} \\ &\leq P \sum_{j=2}^{\infty} \mathbb{E} \left( (27U_j^2 + 2) \left(\frac{\sqrt{6}}{3}U_jR_c - r_0\right)^{-\alpha} \right) \\ &= P \sum_{j=2}^{\infty} \int_{j-1/2}^{j+1/2} (27x^2 + 2) \left(\frac{\sqrt{6}}{3}xR_c - r_0\right)^{-\alpha} dx \\ &= \frac{9P \left( \left(\frac{251}{6}\alpha^2 - \frac{769}{6}\alpha + 89\right) R_c^2 - 27\sqrt{6}R_cr_0(\alpha-1) + 54r_0^2 \right)}{2\sqrt{6}(\alpha-1)(\alpha-2)(\alpha-3)R_c^3 \left(\frac{\sqrt{6}}{2}R_c - r_0\right)^{\alpha-1}} \quad (14) \end{aligned}$$

Substituting (14) into (13), Lemma 1 is proved.

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